PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 30
Adiabatic theorem

November 4

Lec 29: action-angle variables (ϕ_A, J_A) w/ $\underbrace{H = H(J_A)}_{t\text{-independent}}$

$\{\phi_A, J_B\} = \delta_{AB}$

If found, then: $\dot{\phi}_A = \{\phi_A, H\} = \frac{\partial H}{\partial J_A} = \omega_A$ (constant)

so $\phi_A(t) = \phi_A(0) + \omega_A(\vec{J})t$

and $\dot{J}_A = -\frac{\partial H}{\partial \phi_A} = 0$ so J_A's are constants of motion.

Finding AA variables $\rightarrow$ system is solvable.

If trajectories bounded, $\phi_A \sim \phi_A + 2\pi$ is periodic.

Today: Adiabatic Thm: given $H(J_A; \lambda(t))$ ← t-dep. parameters

at each t, we could find AA vars w/ const. λ.

Then if $\lambda(t)$ changes slowly enough, $\dot{J}_A \approx 0$.

J_A is an adiabatic invariant.

How slowly does λ have to change for thm to hold?

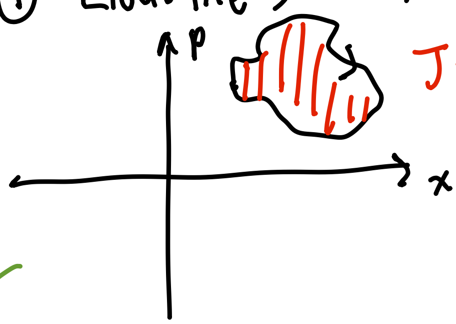
$$\frac{\dot{\lambda}}{\lambda} \ll \omega_1, \ldots, \omega_n \qquad \left(\omega_A = \frac{\partial H}{\partial J_A}\right).$$

τ "time scale over which λ varies"

these are the only time scales in "unperturbed" H.

Proof sketch:

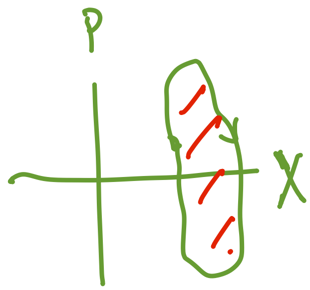
① Liouville's Thm:



$J = \frac{1}{2\pi}\,\text{Area} = \frac{1}{2\pi}\oint p\,dx$

↳ Stoke's Thm: $J = \frac{1}{2\pi}\int dx\,dp\ \omega_{px}$ $\overset{\shortparallel}{1}$

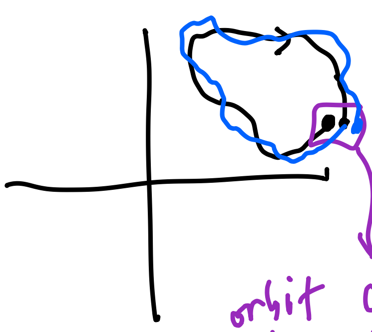
Canonical transformations would leave PBs & ω_{px} invariant...



$J' = \frac{1}{2\pi}\,\text{Area}' = \frac{1}{2\pi}\int dX\,dP\ \omega_{PX}$ $\overset{\shortparallel}{1}$

Liouville's Thm: ω invariant under CTs, so is Area ... & J

② Now let λ vary on time scale $\tau \gg \omega^{-1}$



CT from $(x, p) \longrightarrow (x(\tau), p(\tau))$

orbit doesn't quite close due to $\dot{\lambda}$, but "almost" does if τ large

Almost closed orbits have equal "Area" by Liouville's Thm.

Thus: $J = \frac{1}{2\pi}\,\text{Area} = \frac{1}{2\pi}\oint p\,dx$ is $\approx$ const. of motion.

Example 1: Harmonic Oscillator $\quad H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$

Lec 29: $H(J) = \omega J \qquad [x \propto \sqrt{J}\sin\phi \;\;\&\;\; p \propto \sqrt{J}\cos\phi]$

Scenario 1: $\omega(t)$ slowly increasing... $\omega(0) = \omega_0$

$\omega(\tau_f) = 2\omega_0$

Question: what is amplitude of oscillations? $|x(\tau_f)|$ [or $\tau_f + \frac{1}{2\omega_0} \cdots$]

Adiabatic Thm: J = const. so

$$\frac{H_f}{\omega_f} = \frac{H_0}{\omega_0} \quad \text{so} \quad H_f = 2H_0 \rightarrow H_0 = \frac{1}{2} m\omega_0^2 x_0^2$$

Now: $H_f = 2H_0 = \frac{1}{2} m\omega_f^2 x_f^2 = \frac{1}{2} m \cdot 4\omega_0^2 x_f^2$ and $x_f^2 = \frac{H_0}{m\omega_0^2}$

Thus $x_f = \frac{x_0}{\sqrt{2}}$.

Scenario 2: At $t \approx 0$, $\omega(t)$ instantaneously doubles.

NOT adiabatic.

Now what remains constant across jump is... (x, p).

Answer to question $(x_f?)$ no longer well-posed.

- If jump occurs at $x = x_0$, $x_f = x_0$
- If jump ... $x = 0$, $x_f = \frac{1}{2} x_0$.

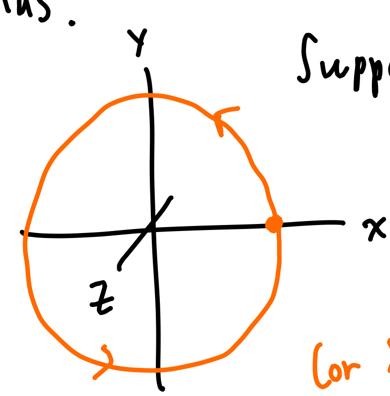
$\Big\}$ adiabatic answer in between

Example 2: particles in magnetic fields.

$$H = \frac{1}{2m}(p_i - qA_i)(p_i - qA_i)$$

$$H = \frac{p_r^2}{2m} + \frac{p_z^2}{2m} + \frac{\left(p_\theta - \frac{q}{2} B_0 r^2\right)^2}{2mr^2}$$

Suppose $\vec{B} = B_0 \hat{z}$



(or helix if $p_z \neq 0$)

p_θ is constant (angular momentum conservation).

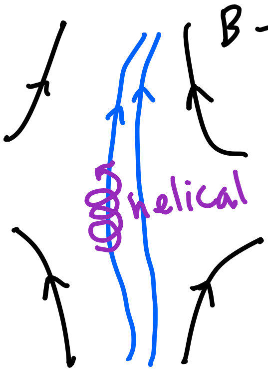
$J_\theta = \frac{1}{2\pi}\oint p_\theta \, d\theta = p_\theta$ is an adiabatic invariant.

The simple circular orbit above: sit at minimum of

$V_{eff}(r) = \frac{1}{2mr^2}\left(p_\theta - \frac{q}{2}B_0 r^2\right)^2$. Minimum at $p_\theta = \frac{qB_0}{2} r^2$.

$J_\theta = \frac{1}{2\pi}(\pi r^2) q B_0$

$= \frac{1}{2\pi} q \, \Phi_B \leftarrow$ magnetic flux thru orbit.



B-field slowly varies

helical orbit w/ $p_z \neq 0$.

↳ Assume z-motion is "adiabatic":

$$\frac{|p_z|}{m} \cdot \frac{\partial_z B_0}{B_0} \sim \frac{\partial_z B_0}{B_0} \ll \frac{qB_0}{m} \quad \text{(cyclotron frequency)}$$

If holds: adiabatic Thm: $\Phi_B \propto J_\theta \approx$ const. of motion.

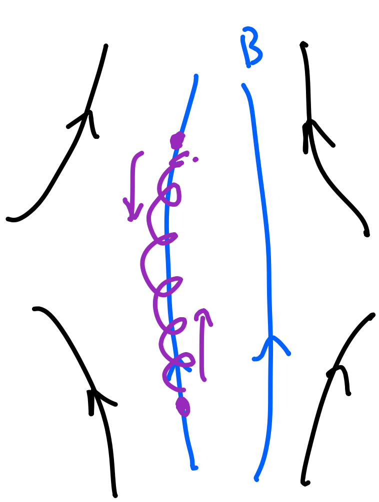
Energy conservation: $E = \frac{1}{2}m(v_z^2 + v_\theta^2) = \text{const.}$

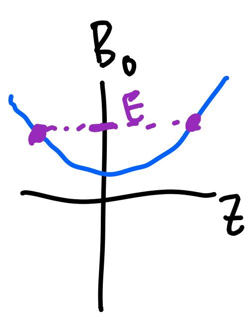
v_z: along B-field. v_θ: orbit

$v_\theta = \omega r = \frac{qB_0}{m} r$ ↳ $E = \frac{m}{2}\left[v_z^2 + \underbrace{\frac{q^2}{m^2}B_0^2 r^2}_{= \frac{qB_0}{m^2}\frac{\Phi_B}{\pi}}\right]$

eff. 1d problem

$B_0(z)$ slowly varies.





Charged particle trapped!

"magnetic mirror"