PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 31
Integrable systems

November 6

Lec 29: action-angle variables: canonical conjugates (ϕ_A, J_A)
such that $\underset{\uparrow\ t\text{-ind.}}{H} = H(\vec{J})$, then

$\dot{J}_A = -\frac{\partial H}{\partial \phi_A} = 0$ and $\dot{\phi}_A = \frac{\partial H}{\partial J_A} = \omega_A$ ← const.

and $\phi_A(t) = \phi_A(0) + \omega_A t$.

Intuitive: integrable if solvable

Precise: Hamiltonian system on 2n-dim phase space is integrable if you can find n independent constants of motion, s.t. $\{I_A, I_B\} = 0$.

$I_n \neq f(I_1, \ldots, I_{n-1})$

Claim: 2-d phase space ⇒ integrable
b/c take $I_1 = H$. ← t-ind. crucial! (Lec 36)

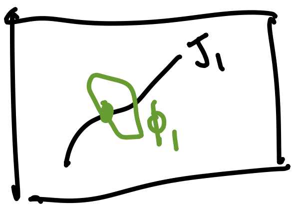
Useful choice for I_A's is often action-angle vars:
$I_A \to J_A$. Since $\{J_A, J_B\} = 0$.

Liouville's Integrability Thm: if I've found n I_A's $\left(\{I_A, I_B\}=0\right)$ and surfaces of const. I_A are compact (don't go to ∞), then $\longrightarrow$ AA vars exist & $\phi_A \sim \phi_A + 2\pi$, <u>locally</u>.

$\hookrightarrow$ geometric decomposition of phase space!

$$\mathbb{R}^{2n} \rightsquigarrow (J_1, \dots, J_n, \phi_1, \dots, \phi_n)$$

$\hookrightarrow$



$\rightsquigarrow$ phase space locally looks like

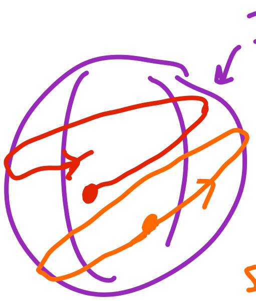
$$\underbrace{M_n}_{J\text{'s}} \times \underbrace{T^n}_{\phi\text{'s}}$$

where $T^n = S^1 \times \dots \times S^1$ (n-dim torus) $\leftarrow$ $\phi_A \sim \phi_A + 2\pi$.

Sketch of proof: Lec 22: function I_1 generates CT along $\boxed{\text{surfaces of const. } I_1}$

$\hookrightarrow$ $\xi^\alpha \to \xi^\alpha + s\{I_1, \xi^\alpha\} + \frac{s^2}{2}\{I_1, \{I_1, \xi^\alpha\}\} + \dots$

By assumptions, surface of const. I_1 was compact



flow stays in bounded region.

Claim: there exists a choice of I_1 w/

smallest "time" $\rightsquigarrow$ $\underbrace{e^{s_0 \cdot \mathrm{ad}_{I_1}} \xi^\alpha = \xi^\alpha}$

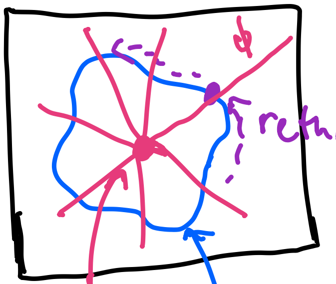
Then flow generated is just $\phi_1 \to \phi_1 + s_0$

Re-scale $I_1 \to J_1$ such that everywhere $e^{2\pi \cdot \mathrm{ad}_{J_1}} \xi^\alpha = \xi^\alpha$.

Iterate procedure: look for I_2, ind. of J_1, w/ same property....
(e.g. symplectic reduction)

Implicit: I_A's at the start were smooth... (return later!)

$n=2$: take $I_1 = H$



ϕ

return to itself after time T

choose ϕ so that $\phi(T) = 2\pi$

$\downarrow$

$J = f(H)$ s.t. $\phi \sim \phi + 2\pi$ holds on all orbits.

fixed pt (max/min)

curve of const. H

LI Thm: "local polar coords" exist for bounded orbits.

↳ but this works in higher d too!

In practice: hard to do this...

→ how to find I_A's?

(Lax equation in 1d chains ... beyond class)

→ high-d ⇝ chaos easier...

→ many systems are close to integrable!

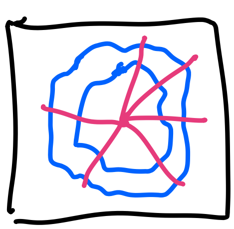
e.g. Solar system: perturbed $\underbrace{\text{Kepler/central force}}_{\text{is integrable!}}$

most field theories are perturbed free theories...

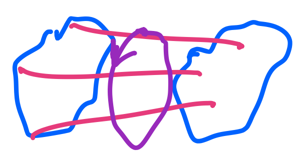
How do we know that I_A's don't exist?

↳ assumption of smoothness... is important! (lec 35)/chaos

↳ $n=2$:



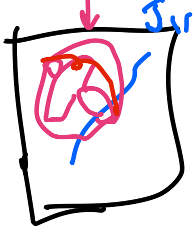
$\rightsquigarrow$



$\dot{J} = 0$ and $\dot{\phi} = \frac{\partial H}{\partial J} = \text{const.}$

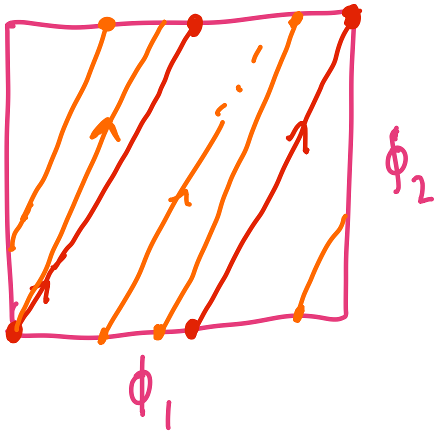
$n=4$: (ϕ_1, ϕ_2)

J_1, J_2



$\dot{\phi}_1 = \omega_1 = \text{const.}$

$\dot{\phi}_2 = \omega_2 = \text{const.}$



$S^1 \times S^1 = T^2 \simeq$ box w/ periodic BCs

commensurate orbit return to starting point at finite t_*

$\phi_1(t_*) = \omega_1 t_* = 2\pi n_1$ ← integer

and $\phi_2(t_*) = \omega_2 t_* = 2\pi n_2$.

$\boxed{\text{Commensurate: } \frac{\omega_1}{\omega_2} = \frac{n_1}{n_2} = \text{rational number.}}$

most orbits are $\boxed{\text{incommensurate: } \frac{\omega_1}{\omega_2} = \text{irrational number}}$

never return to starting point.

But get arbitrarily close... to every point on T^2.

Incommensurate orbit "uniformly averages" over (ϕ_1, ϕ_2) in time.

↳ "ergodicity": $\frac{1}{T}\int_0^T dt\, f \to \int \frac{d\phi_1 d\phi_2}{(2\pi)^2} f$

↓

"math fact": irrational α:

$p + \alpha q$ can be arbitrarily close to any #.

↑ ↗ integers

Same ideas in higher dim: some pairs of ω_1, ω_2 might be com. while ω_2, ω_3 are incom...

Key observation: depending on # of commensurate freq.,
phase space trajectories seem to fill $1, 2, \ldots, n$-dim
subspace of phase space.

Claim (much later): not true in chaos.
- $> n$-dim.
- dim does not need to be integer (fractal)