PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 32
Perturbation theory: one degree of freedom

November 8

Start w/ integrable system $H_0(\vec{J})$

↳ have AA variables!

Perturb: $H(\vec{J},\vec{\Phi}) = H_0(\vec{J}) + \varepsilon H_1(\vec{J},\vec{\Phi})$. What happens?

Today: 1 DOF (2d phase space) → integrable anyway.

Example: $H = \underbrace{\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2}_{H_0} + \underbrace{\varepsilon x^4}_{H_1}$ ↗ taking ε to be perturbatively small.

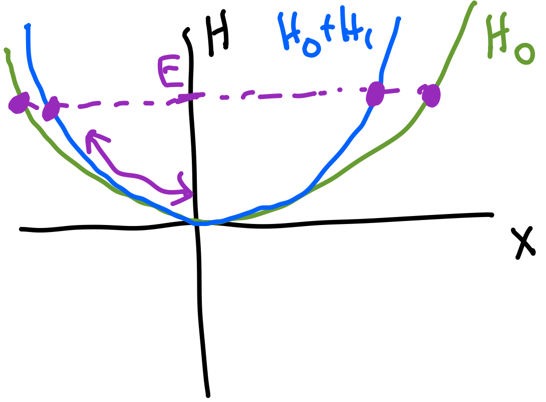
If H_1 perturbative, how small is ε?

$H_1 \ll H_0$. Suppose oscillation of amplitude $x \sim A$.

↳ $\varepsilon A^4 \ll \frac{1}{2}m\omega^2 A^2 \sim E$ ⤳ $A^2 \ll \frac{m\omega^2}{\varepsilon}$

or $E \ll \frac{(m\omega^2)^2}{\varepsilon}$

What happens if E is this small?



dynamics still oscillates.

↳ system is still integrable...

Method 1: Literal Interpretation.

$$x(t) = x_0(t) + \varepsilon x_1(t) + \dots \qquad H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \varepsilon x^4$$

$$p(t) = p_0(t) + \varepsilon p_1(t) + \dots$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad \dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x - 4\varepsilon x^3$$

↳ $\dot{x}_0 + \varepsilon \dot{x}_1 + \dots = \frac{p_0}{m} + \varepsilon \frac{p_1}{m} + \dots$

↳ $\dot{p}_0 + \varepsilon \dot{p}_1 = -m\omega^2(x_0 + \varepsilon x_1) - 4\varepsilon(x_0 + \varepsilon x_1)^3$

Collect terms at $\mathcal{O}(\varepsilon^0)$:

$$\dot{x}_0 = \frac{p_0}{m} \quad \text{and} \quad \dot{p}_0 = -m\omega^2 x_0$$

Solve: $x_0 = A\cos(\omega t)$ and $p_0(t) = m\omega A\sin(\omega t)$

Terms at $\mathcal{O}(\varepsilon)$: $x_1(0) = p_1(0) = 0.$ sources a perturbation...

$$\dot{x}_1 = \frac{p_1}{m} \qquad \dot{p}_1 = -m\omega^2 x_1 - \boxed{4x_0^3}$$

↳ $x_1(t) = \frac{3A^3}{8\omega^2}\left[\sin(3\omega t) - 7\sin(\omega t) + \underbrace{4\omega t\cos(\omega t)}\right]$

diverges as t → ∞.

So PT failed. !! $(x_1 \gg x_0)$

Why? $x(t) \sim (A_0 + \varepsilon A_1)\cos((\omega_0 + \underbrace{\varepsilon\omega_1)t} + \varepsilon t_1))$

$x_1(t) \sim \omega_1 t \sin(\omega_0 t) \dots$

qualitatively similar

Method 2: transform from old → new AA variables.

$$(J_0, \phi_0) \qquad (J, \phi)$$

Find a CT such that $H_0(J_0) + \varepsilon H_1(J_0, \phi_0) \longrightarrow H(J)$

↳ Find Type 2 generating function: (lec 27)

$$S(\phi_0, J) = \phi_0 J + \varepsilon S_1 + \varepsilon^2 S_2 + \cdots$$

↳

$$J_0 = \frac{\partial S}{\partial \phi_0} = J + \varepsilon \frac{\partial S_1}{\partial \phi} + \cdots$$

$$\phi = \frac{\partial S}{\partial J} = \phi_0 + \varepsilon \frac{\partial S_1}{\partial J} + \cdots$$

$$H(J) = H_0(J_0) + \varepsilon H_1(J_0, \phi_0)$$

$$= H_0\left(J + \varepsilon \frac{\partial S_1}{\partial \phi} + \cdots\right) + \varepsilon H_1\left(J + \varepsilon \frac{\partial S_1}{\partial \phi} + \cdots, \phi_0\right).$$

At order $O(\varepsilon^0)$: $\quad H(J) \approx H_0(J) + \cdots$

At order $O(\varepsilon)$: $\quad H \approx H_0(J) + \left[\underbrace{\frac{\partial H_0}{\partial J}}_{\text{call this } w_0(J)} \frac{\partial S_1}{\partial \phi_0} + H_1(J, \phi_0)\right]\varepsilon + \cdots$

The left hand side should only depend on J.

Choose S_1 to make this happen!

Brute force: write:

$$H_1(J, \phi_0) = \sum_{m=-\infty}^{\infty} e^{im\phi_0} h_m(J) \quad \& \quad S_1(J, \phi_0) = \sum_{m=-\infty}^{\infty} e^{im\phi_0} s_m(J).$$

$$H(J) = H_0(J) + \varepsilon\Big[\underbrace{w_0 \sum_m im\, e^{im\phi_0} s_m + \sum_m e^{im\phi_0} h_m}_{\varepsilon \sum_m e^{im\phi_0}\big[\underbrace{im w_0 s_m + h_m}_{\text{must vanish if } m \neq 0}\big]}\Big].$$

So choose: $S_m = -\dfrac{h_m}{im\omega_0}$ if $m \neq 0$. (Can take $s_0 = 0$).

Then: new $H(J) = H_0(J) + \varepsilon h_0(J) + \cdots$

Find h_0: $h_0(J) = \displaystyle\int_0^{2\pi} \frac{d\phi_0}{2\pi} \cancel{e^{-i0\cdot\phi_0}}{}^{1} H_1(J, \phi_0)$

$\sim$ t-avg of perturbation H_1 (cf. ergodicity lec 31)

Most important result:

perturbed frequencies: $\dfrac{\partial H}{\partial J} = \omega_0(J) + \varepsilon \dfrac{\partial h_0}{\partial J} + \cdots$

→ generally not 0...

↓ cures old problem. PT well-behaved (here).

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Example again: $H = \underbrace{\dfrac{p^2}{2m} + \dfrac{1}{2}m\omega^2 x^2}_{H_0} + \underbrace{\varepsilon x^4}_{H_1}$

lec 29: convert to AA variables: old (J_0, ϕ_0)

$$H = \omega J_0 + \varepsilon\left(\sqrt{\frac{2J_0}{m\omega}}\sin\phi_0\right)^4$$

Calculate $h_0(J) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} d\phi_0 \left(\frac{2J_0}{m\omega}\right)^2 \sin^4\phi_0 = \frac{3}{2}\frac{J^2}{m^2\omega^2}$

New Hamiltonian $H(J) = \omega J + \varepsilon \cdot \dfrac{3}{2}\dfrac{J^2}{m^2\omega^2} + \cdots$

Oscillation frequency: $\omega(J) = \dfrac{\partial H}{\partial J} = \omega + 3\varepsilon\dfrac{J}{m^2\omega^2} + \cdots$

$$\to \omega(E) \approx \omega + \frac{3E\varepsilon}{m^2\omega^3} + \cdots$$

When will PT fail? $\mathcal{O}(\varepsilon)$ term is comparable to $\mathcal{O}(\varepsilon^0)$...

when $E \gtrsim \dfrac{m^2\omega^4}{\varepsilon}$, confirms earlier intuition. ☺