PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 36
Kicked rotor

November 18

Lec 35: t-ind H: ≥ 2 DOF (≥ 4d phase space) for chaos

Today: t-dep. H: 1 DOF (2d phase space) w/ chaos

Kicked rotor: canonical conjugates $\{\phi, J\} = 1$. $\phi \sim \phi + 2\pi$

$$H(t) = \frac{J^2}{2I} + \sum_{n=-\infty}^{\infty} \delta(t - n\tau)\, \varepsilon_0 \cos\phi$$

EOMs:

$$\dot{\phi} = \frac{\partial H}{\partial J} = \frac{J}{I}$$

$$\dot{J} = -\frac{\partial H}{\partial \phi} = \underbrace{\sum_{n=-\infty}^{\infty} \delta(t - n\tau)\, \varepsilon_0 \sin\phi}$$

$\dot{J} = 0$ if $t \neq n\tau$ (free rotor)
due to δ functions,
$J(t)$ will be piecewise constant.

Goal: integrate equations between kicks ($0 < t \leq \tau$).
Suppose just after kick $t=0$: $\phi(0^+) = \phi_0$ & $J(0^+) = J_1$

Since $\dot{J} = 0$ for $0 < t < \tau$:

$J(t) = J_1$ for $0 < t < \tau$.

$\dot{\phi}(t) = \frac{J}{I}$ means for $0 < t < \tau$: $\phi(t) = \phi_0 + t\frac{J_1}{I}$

Just before $t = \tau$: $J(\tau^-) = J_1$ & $\phi(\tau^-) = \underbrace{\phi_0 + \frac{\tau}{I} J_1}_{\text{call } \phi_1}$ ①

So $\phi_n = \phi(n\tau^-)$ and $J_n = J(n\tau^-)$

What happens just after kick?

$$\underset{\delta \to 0}{J(\tau+\delta)} = J(\tau-\delta) + \int_{\tau-\delta}^{\tau+\delta} dt\, \dot{J}(t) = J(\tau-\delta) + \int_{\tau-\delta}^{\tau+\delta} dt \underbrace{\sum_{n=-\infty}^{\infty} \delta(t-n\tau)}\, \varepsilon_0 \sin\phi(t)$$

$$\int_{\tau-\delta}^{\tau+\delta} \delta(t - 1\cdot\tau)\, \varepsilon_0 \sin\phi(t) = \varepsilon_0 \sin\phi_1$$

$$J_2 = J_1 + \varepsilon_0 \sin\phi_1 \quad ②$$

Why is $\phi(\tau^+) = \phi(\tau^-) = \phi_1$?

$J(t)$ not divergent

$$\phi(\tau+\delta) = \phi(\tau-\delta) + \int_{\tau-\delta}^{\tau+\delta} dt\, \dot{\phi}(t) = \phi(\tau-\delta) + \underbrace{\int_{\tau-\delta}^{\tau+\delta} dt\, \frac{J(t)}{I}}_{\leq 2\delta \frac{1}{I} \max(J) \to 0 \text{ as } \delta \to 0}$$

Iterate arguments. Since $\phi_n = \phi(n\tau^-)$ & $J_n = J(n\tau^-)$,

② $\rightarrow$ $J_{n+1} = J_n + \varepsilon_0 \sin\phi_n$

① $\rightarrow$ $\phi_{n+1} = \phi_n + \frac{\tau}{I} J_{n+1} = \phi_n + \frac{\tau}{I} J_n + \frac{\tau\varepsilon_0}{I} \sin\phi_n$

Started w/ continuous t... reduced to discrete map (discrete time)

As before, work in dimensionless units:

$[\phi] = \text{const.}$

$[t] = [T]$

$[J] = [M][L]^2[T]^{-1}$

$[\tau] = [T]$

$[I] = [M][L]^2$

$[\varepsilon_0] = [J] = [M][L]^2[T]^{-1}$

$\leftarrow$ from (2)

Define:

$\tilde{\phi}_n = \phi_n$ and $\tilde{J}_n = \frac{\tau}{I} J_n$

Notice

$\frac{\tau \varepsilon_0}{I}$ = dimensionless! $= \varepsilon$

Kicked rotor map:

$$\left.\begin{aligned} \tilde{J}_{n+1} &= \tilde{J}_n + \varepsilon \sin \tilde{\phi}_n \\ \tilde{\phi}_{n+1} &= \tilde{\phi}_n + \tilde{J}_{n+1} \end{aligned}\right\}$$

Recall: $\tilde{\phi}_n \sim \tilde{\phi}_n + 2\pi$

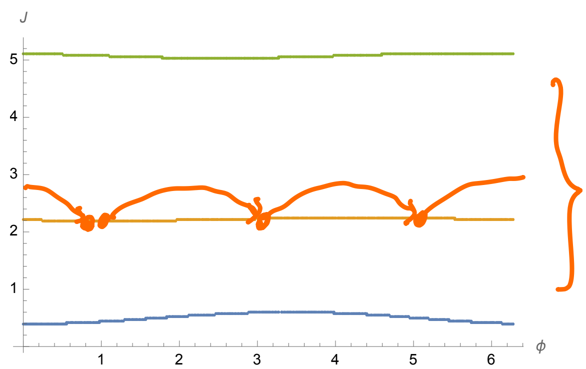
If also "identify" $\tilde{J}_n \sim \tilde{J}_n + 2\pi$...

map also same.

Now, drop tildes:

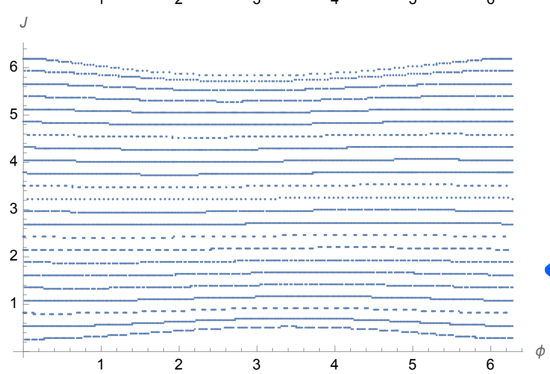
$\tilde{\phi} \to \phi$ & $\tilde{J} \to J$.

Numerical sims at $\varepsilon = 0.05$



plot of 3 different ICs.

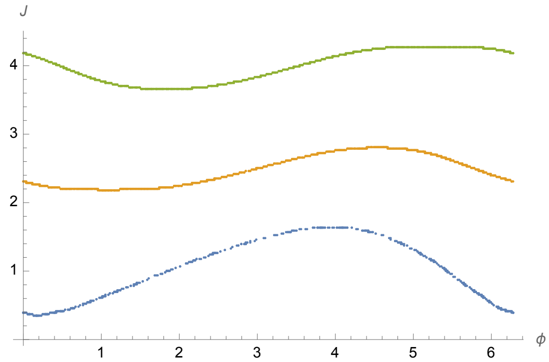
it's a discrete map... plotting (ϕ_n, J_n)



20 diff ICs.

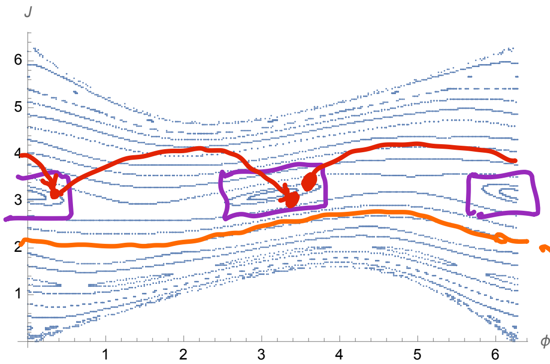
Remind of integrability:

only exploring 1 dimension out of 2.



Numerics at $\varepsilon = 0.6$

still looks integrable?

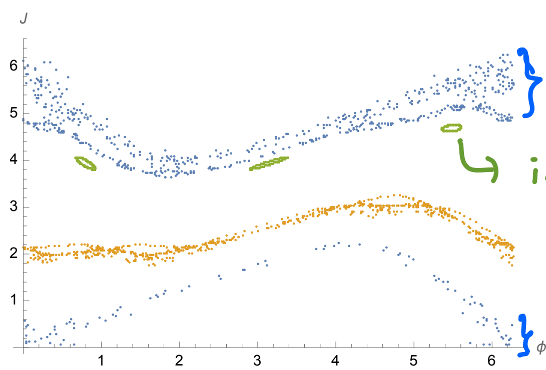


near $J = \pi$ … $\phi_{n+2} \approx \phi_n$?

"islands" → still integrable.

trajectory still "winds around phase space"

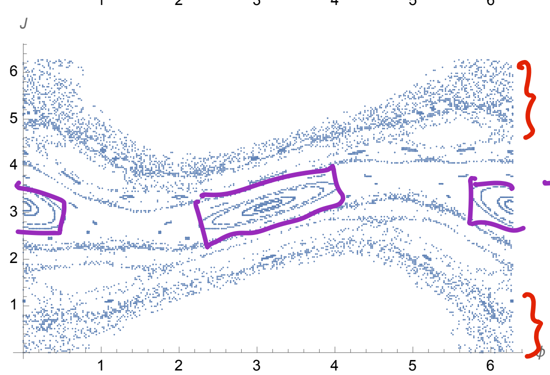
cf. lec 33.



Numerics at $\varepsilon = 1.05$

integrable still

fill out 2d? chaos…



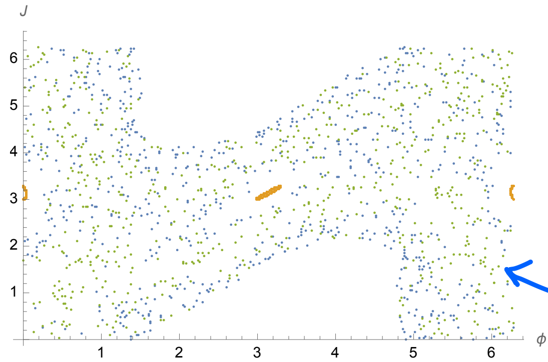
lose track of the IC associated w/ each point

$J = \pi$ islands still there…

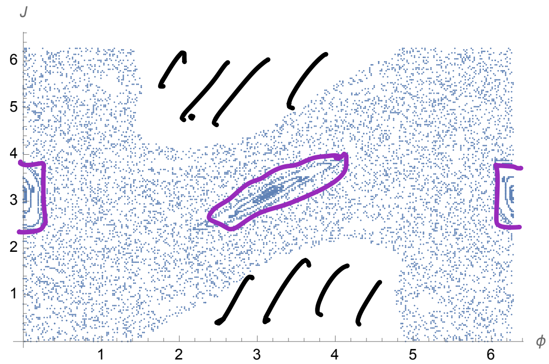
~ butterfly effect of chaos (lec 40)

"Onset of chaos" at $\varepsilon_c \sim 0.95$

Numerics at $\varepsilon = 1.8$



2 diff IC mix $\rightsquigarrow$ chaos



islands of integrability getting weaker

$\phi_0 = 0$ for all ICs