PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 37
Logistic map

November 20

Chaotic systems:

① Hénon-Heiles
t-ind. Hamiltonian
4D Phase space

② kicked rotor
t-dep. Hamiltonian
$\underbrace{\text{2D phase space}}$
↓
discrete map
$J_{n+1} = f(J_n, \phi_n)$

→ ③ logistic map:
chaotic map for one variable.
NOT Hamiltonian
↳ dissipative

Logistic map: $x_{n+1} = f(x_n) = \overset{\text{const.}}{r}\, x_n(1-x_n)$

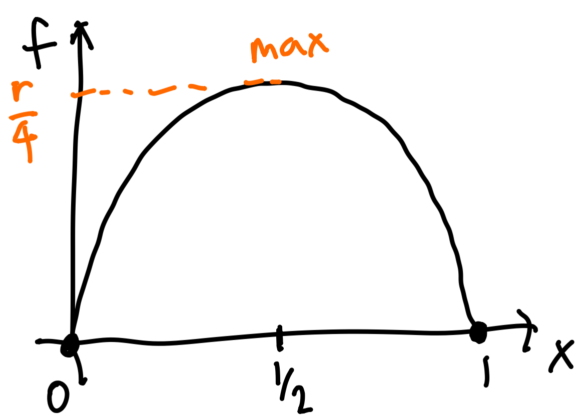
What is behavior of $\{x_0, x_1, \ldots, x_n, \ldots\}$

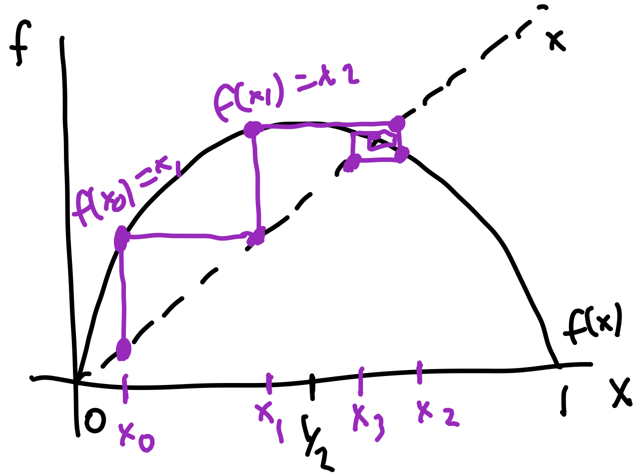
Demand: $\underbrace{0 \le x_n}_{} \le 1$

$r > 0$

So if $f(x_n) \le 1$
↑ including $f(\frac{1}{2})$

need $0 \le r \le 4$





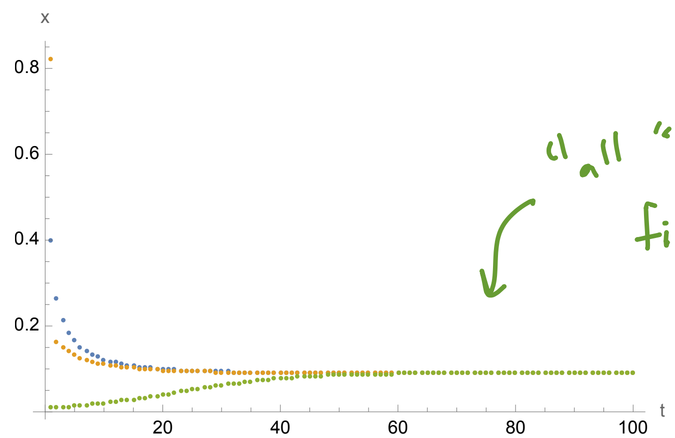
"cobweb diagram"

For this value of r, dynamics approaching fixed point: $x_* = f(x_*)$. ← stable

As before ⟶ numerical simulation

First: 3 different ICs at same r

$r = 1.1$

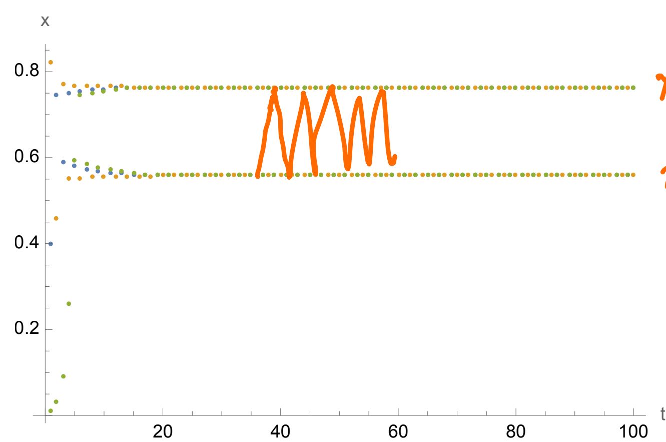


"all" trajectories approach same fixed point ← stable

$$x_* = f(x_*) = r x_* (1 - x_*)$$

$x_* = 0$ or $\boxed{x_* = 1 - 1/r.}$

$r = 3.1$



$\left.\begin{array}{l} x_{*2} \\ x_{*1} \end{array}\right\}$ stable period-2 cycle:

$f(x_{*1}) = x_{*2}$

$f(x_{*2}) = x_{*1}$

Definition: period-n cycle has

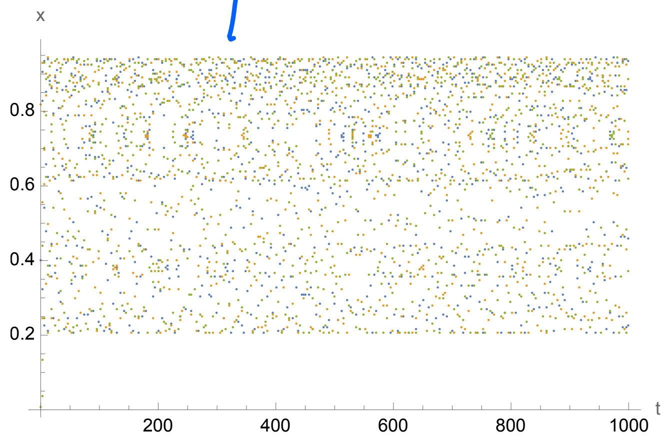
$$x_{*1} \xrightarrow{f} x_{*2} \xrightarrow{f} \cdots \xrightarrow{f} x_{*n} \xrightarrow{f} x_{*1}$$

$f(x_{*1}) = x_{*2}$

all ICs explore "a lot" of phase space

chaos: initial condition tells us little about x_n... (butterfly effect)

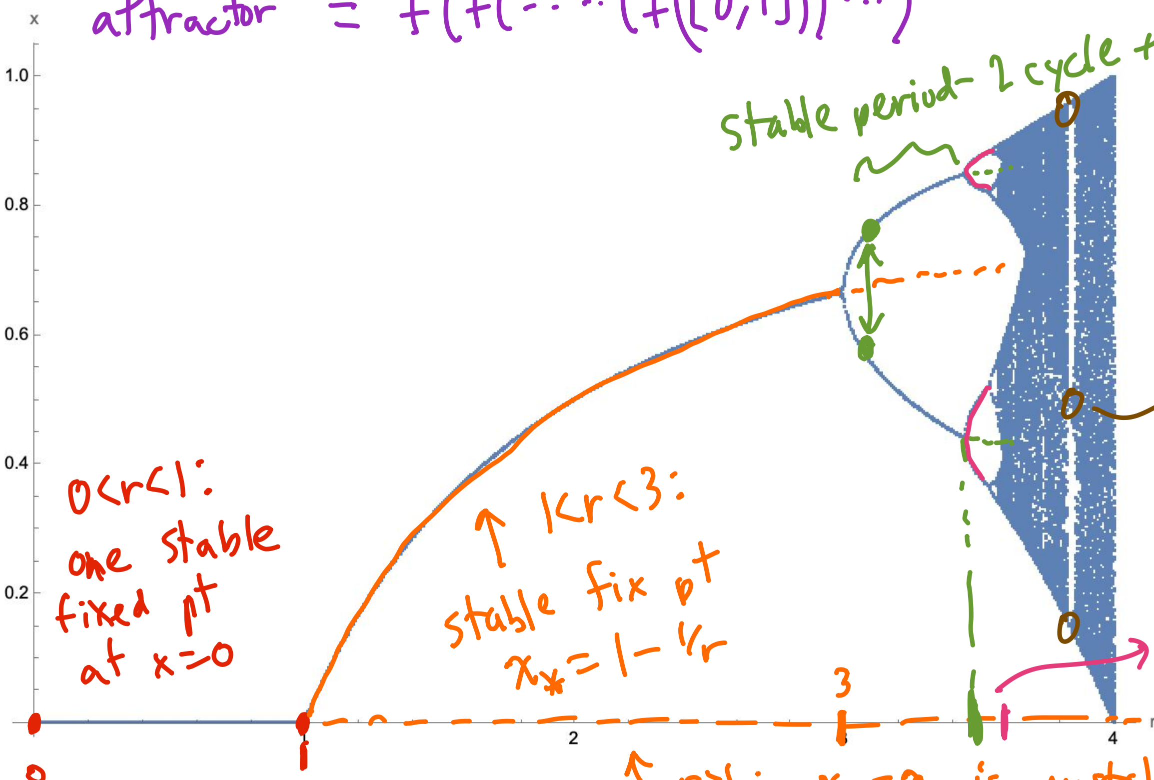
$r = 3.77$



Note: dissipative map, less clear how to properly "dimension count"

Plot of "large n" values of x_n for various ICs

attractor $= f(f(\cdots(f([0,1]))\cdots))$



stable period-2 cycle + 2 unstable fix pts

period-3 cycle amidst chaos (!)

$0<r<1$: one stable fixed pt at $x=0$

$1<r<3$: stable fix pt $x_* = 1 - 1/r$

stable period-4 cycle

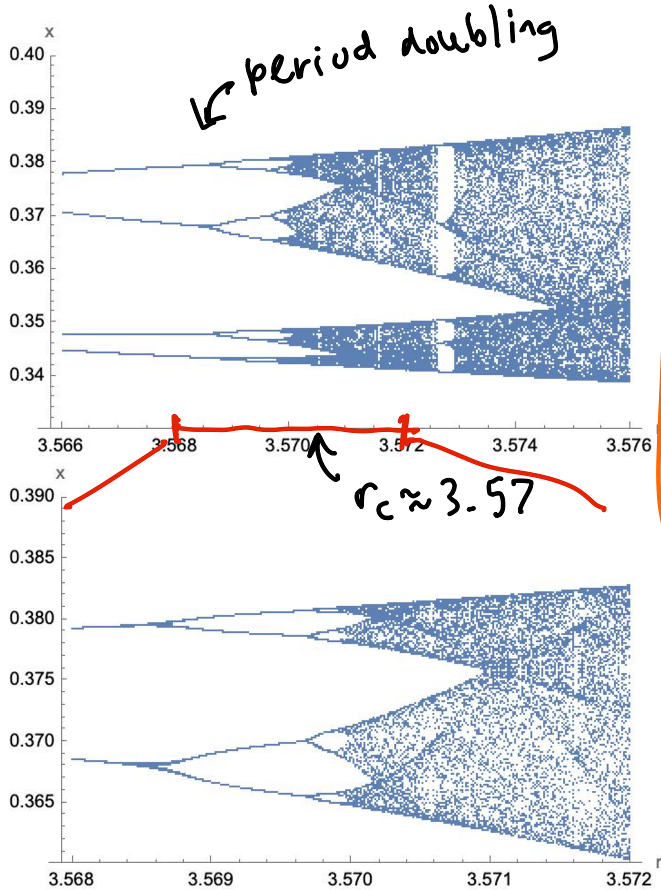
$r>1$: $x_* = 0$ is unstable

HW12: $r=4$ = chaos + exact solvability!

Onset of chaos:

NOT whole attractor

period doubling



$r_c \approx 3.57$

Attractor "geometry" has "self-similar" structure near r_c.

(w/o axis labels you can't deduce scale)

$\Rightarrow$ fractal (Lec 41), object w/ non-integer dimension

structure at all scales...

General lessons (for 1d dissipative maps...)

$\rightarrow$ onset of chaos: period doubling "cascade" (Lec 39)

$\rightarrow$ fractal structure of attractor $\rightarrow$ quantitatively universal

self-similar