PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 38
Linear stability analysis

November 22

Logistic map: $x_{n+1} = r x_n (1-x_n)$ $\quad$ $0 \leq x_n \leq 1$

$\uparrow$ $0 \leq r \leq 4$

Minimal model for chaos.

Today: analytic understanding of some r?

Begin: $0 < r < 1$. Claim: $x_n \to 0$ as $n \to \infty$.

Proof: $\frac{x_{n+1}}{x_n} = r\underbrace{(1-x_n)}_{\leq 1} \leq r < 1$. So $x_n, x_{n+1}, x_{n+2} \ldots$ decreasing.

Actually: $x_n \leq r x_{n-1} \leq \cdots \leq r^n x_0 \leq r^n$.

Approach $\underbrace{x_* = 0}_{\text{fixed point}}$ exponentially fast: $x_n \sim \exp\left[-n \log \frac{1}{r}\right]$

$\hookrightarrow$ $x_* = r x_* (1-x_*)$: for any r, $x_* = 0$ is fixed pt.

stable

General theory (linear stability analysis for discrete maps):

Assume fixed point $x_* = f(x_*)$ for $x_{n+1} = f(x_n)$

Linear stability near fixed point:

$x_n = x_* + \delta x_n$ ← infinitesimal, work to first order in δx_n.

assumed Taylor series exist...

$x_{n+1} = f(x_n)$

$x_* + \delta x_{n+1} = f(x_* + \delta x_n) \approx x_* + \underbrace{f'(x_*)}_{\text{call this } \lambda} \delta x_n + \cdots$

↳ $\delta x_{n+1} \approx \lambda \, \delta x_n$.

Therefore $\delta x_n = \lambda^n \delta x_0$.

Stable fixed point:

$\lim_{n\to\infty} |\delta x_n| = 0$

$\boxed{|\lambda| < 1}$

unstable fixed point:

$\lim_{n\to\infty} |\delta x_n|$ "$= \infty$".

↑ Taylor series/ linear analysis fails.

$\boxed{|\lambda| > 1}$

↳ $\lambda = \pm 1$: marginal fixed pt: need to go to higher orders...

How to use linear stability to learn a lot...

① Find fixed points.

$f(x_*) = r x_*(1 - x_*) = x_*$. Solved by:

$x_* = 0$

$\underbrace{x_* = 1 - 1/r}_{\text{only valid for } r > 1}$.

② Analyze stability:

$\lambda = f'(x_*) = r(1 - 2x_*)$

$\lambda = r$:

stable at $r<1$

unstable at $r>1$

marginally stable at $r=1$:

$$\frac{x_{n+1}}{x_n} = 1 - x_n < 1 \quad \text{if } x_n \neq 0.$$

$\lambda = r(1 - 2 + \frac{2}{r}) = 2 - r$

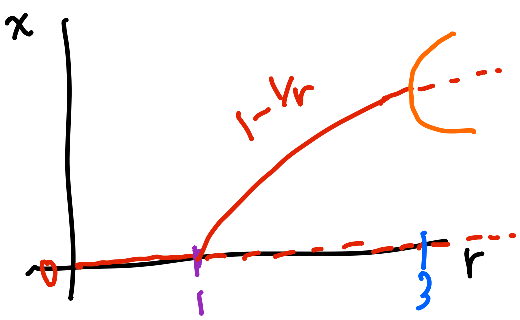
stable for $1<r<3$

unstable for $r>3$

$|\lambda| = |2-r|$

$=1$ at $r=1$ $\quad =-1$ at $r=3$.

So far, guess at the $\underbrace{\text{shape of attractor}}_{\text{all pts that } x_n \text{ approach at large } n}$:



solid = stable

dashed = unstable

Numerical (Lec 37): r (a bit) larger than 3:

stable period-2 cycle

Analytically: define n^{th} iterated map $f^{[n]}(x)$:

$$f^{[n]}(x) = f\left(f^{[n-1]}(x)\right) = \underbrace{f(f(\cdots(f(x)))}_{n \text{ times}}.$$

So: $x_{m+n} = f^{[n]}(x_n)$.

If period-2 cycle which is stable: 2 stable fixed pts:

$$f^{[2]}(x_{*1,2}) = x_{*1,2} \qquad \hookrightarrow \left|f^{[2]\prime}(x_{*1,2})\right| < 1.$$

Goal: for logistic map, find $f^{[2]}$ / fix pts / stability.

$$x_* = r\left[r x_*(1-x_*)\right]\left[1 - r x_*(1-x_*)\right] \qquad \text{i.e.} \quad x_* = f^{[2]}(x_*).$$

Quartic !! but we know 2 solutions:

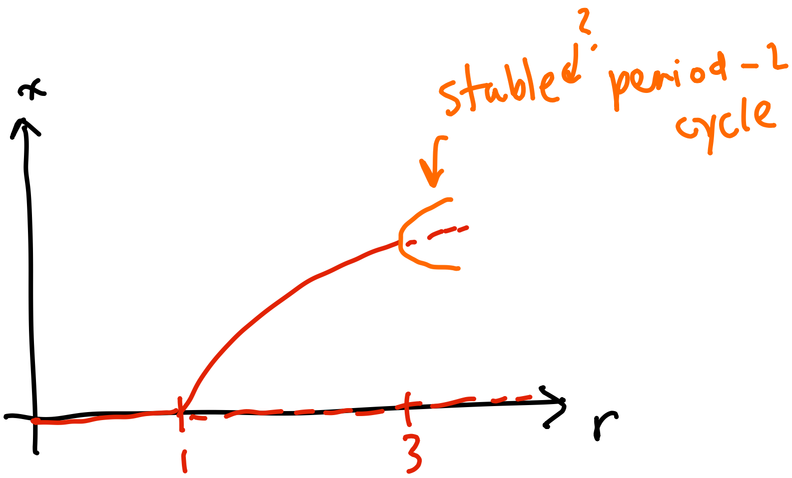
$x_* = 0$ & $x_* = 1 - 1/r$ are fixed points of $f^{[1]}$ AND $f^{[2]}$.

$$0 = \frac{x_* - f^{[2]}(x_*)}{x_*\left(x_* - 1 + 1/r\right)} \xrightarrow{\text{Mathematica}} 0 = r^2 x_*^2 - r(1+r)\,x_* + (1+r)$$

Quadratic formula:

$$x_{*1,2} = \frac{r(1+r) \pm \sqrt{r^2(1+r)^2 - 4r^2(1+r)}}{2r^2} = \underbrace{\frac{1+r \pm \sqrt{(1+r)(r-3)}}{2r}}_{\text{only real if } r>3 \;!!}$$

period doubling (lec 37+39)



If $r = 3 + \varepsilon$:

$$x_{*1,2} = \underbrace{\frac{2}{3} \pm \frac{\sqrt{\varepsilon}}{3}}$$

$= 1 - 1/r$ at $r = 3$
(fix pt that went unstable)

Check stability of period-2 cycle!

$\lambda = \left| f^{[2]\prime}(x_{*1}) \right|$ ← use either point, both go unstable together...

$\hookrightarrow |\lambda| = 4 + 2r - r^2$ (Mathematica).

$+1 = 4 + 2r - r^2$

$r^2 - 2r - 3 = 0$

$r = 1 \pm 2$

$\boxed{r = 3}$

$-1 = 4 + 2r - r^2$

$r^2 - 2r - 5 = 0$

$r = 1 \pm \sqrt{6}$

↓ valid

$r = 1 + \sqrt{6} \approx 3.45$

So: stable period-2 cycle for $3 < r < 3.45\ldots$

For $r > 3.45$, period-2 cycle also unstable ⟶ period-4 ...