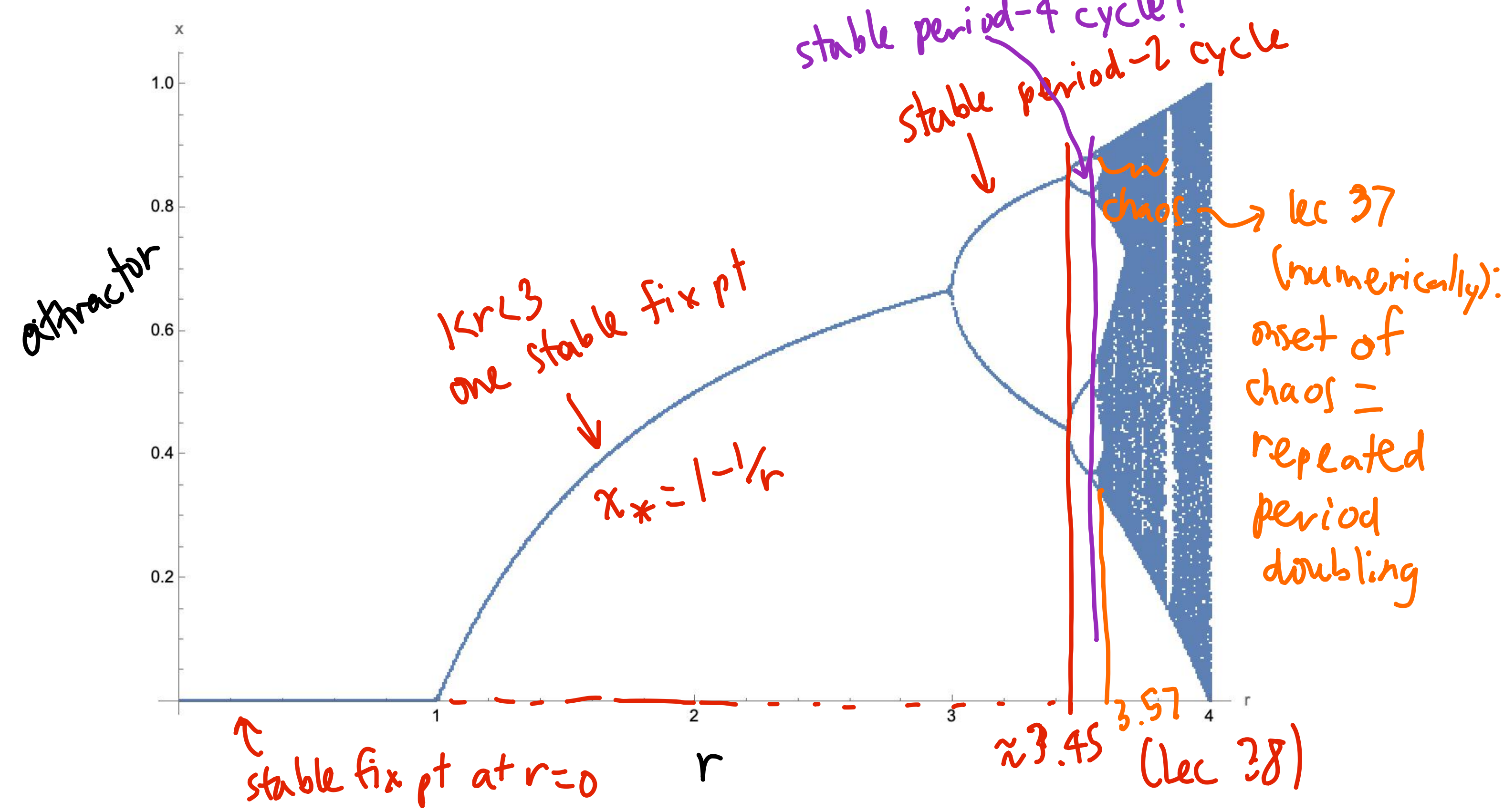
PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 39
Period doubling

December 2

Logistic map: $x_{n+1} = r x_n (1 - x_n)$ $\quad 0 \le x_n \le 1$, $\quad 0 \le r \le 4$

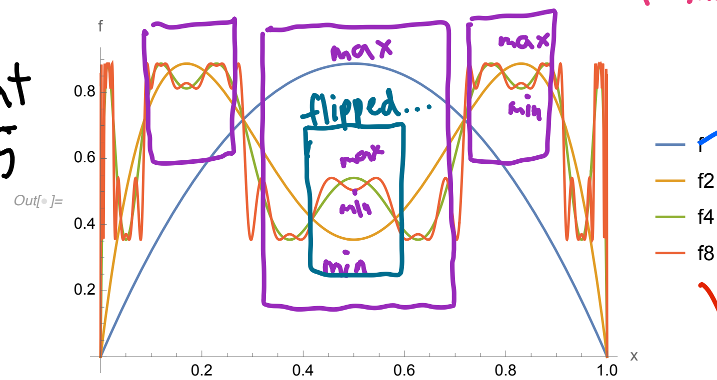
Attractor: set of points x_n (large $n \to \infty$) starting from generic x_0.



Recall: iterated map $f^{[n]}(x) = \underbrace{f(f(\cdots(f(x))\cdots)}_{n \text{ times.}}$

$$f^{[n]}(x_k) = x_{k+n}$$

Look at $r = 3.55$



$\to f^{[1]} = rx(1-x)$ (2^{nd} order)

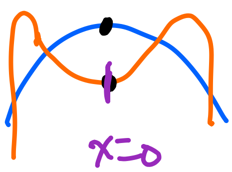
$\to f^{[2]} = f(f(x))$ (4^{th} order)

$\to f^{[4]} = f^{[2]}(f^{[2]}(x))$ (16^{th} order)

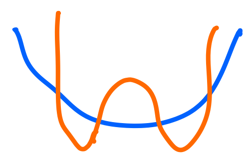
$\to f^{[8]}$ (256^{th} order)

Just before onset of chaos, (by eye) self-similar behavior??

Go from $f^{[2^k]}(x) \to f^{[2^{k+1}]}(x) = f^{[2^k]}(f^{[2^k]}(x))\ldots$



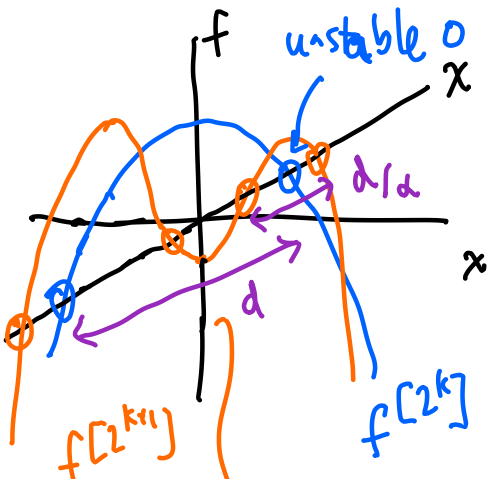
or



← self-similar?

Guess! Put maximum in $f^{[2^k]}$ at $x = 0$.

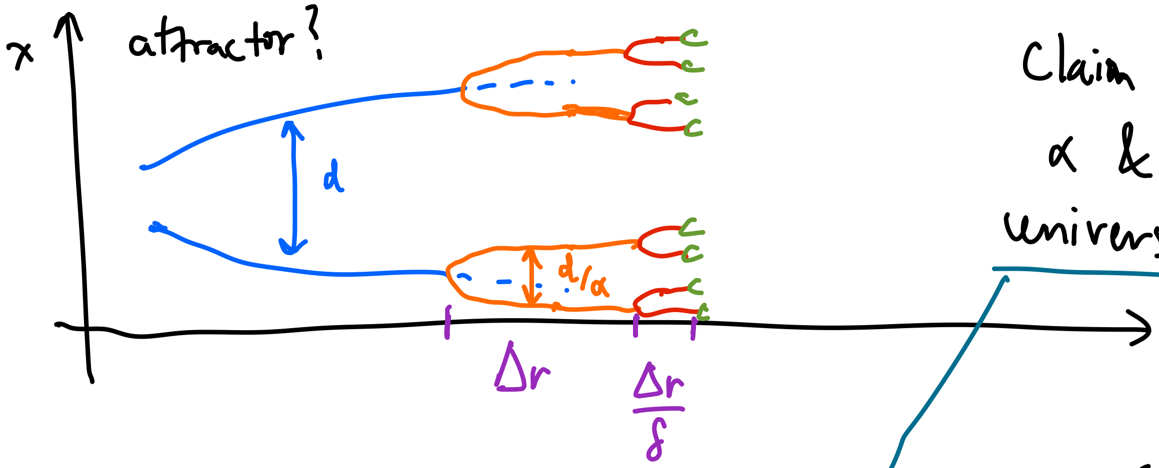
$$f^{[2^{k+1}]}(x) \simeq -\frac{1}{\alpha} f^{[2^k]}(-\alpha x) \quad \text{for some } \alpha > 1$$



at onset of chaos, also unstable....

At higher iterations, more unstable fixed points...

↳ self-similar ~ fractal (lec 41)



Claim (Feigenbaum): α & δ are universal!

characterize self-similarity of attractor at $r \approx r_c \approx 3.57$

Independent of specific choice $f(x) = rx(1-x)$

Associated w/ period-doubling route to chaos.

Conjecture: renormalization group: rescaled & iterated $f^{[2^k]}$ tend to universal function (after rescaling):

$$\lim_{n\to\infty} (-\alpha)^n f^{[2^n]}\left(\frac{x}{(-\alpha)^n}\right) = g(x)$$

1) limit exists (at $r=r_c$) ...

2) $g(x)$ doesn't depend on $f(x) = rx(1-x)$ ... $g(x)$ encodes Feigenbaum numbers.

Use conjecture to estimate $g(x)$ & α:

$$f^{(2^{n+1})}(x) \approx -\frac{1}{\alpha} f^{(2^n)}(-\alpha x)$$

$$f^{(2^n)}(x) = -\alpha f^{(2^{n+1})}\left(-\frac{x}{\alpha}\right) = -\alpha f^{(2^n)}\left(f^{(2^n)}\left(-\frac{x}{\alpha}\right)\right)$$

RG conj.

$$\frac{1}{\cancel{(-\alpha)^n}} g(\underbrace{(-\alpha)^n x}_{z}) = \frac{-\alpha}{\cancel{(-\alpha)^n}} g\left(\cancel{(-\alpha)^n} \cdot \frac{1}{\cancel{(-\alpha)^n}} g\left(-\frac{1}{\alpha} \cdot \underbrace{(-\alpha)^n x}_{z}\right)\right)$$

$$g(z) = -\alpha\, g\left(g\left(-\frac{z}{\alpha}\right)\right).$$

Solve equation for both $g(z)$ & α.

Approximately!

But first, if $g(z)$ is sol'n, then $\lambda \cdot g\left(\frac{z}{\lambda}\right)$ is also sol'n:

$$\lambda g\left(\frac{z}{\lambda}\right) = -\alpha\lambda\, g\left(\frac{1}{\cancel{\lambda}} \cdot \cancel{\lambda}\, g\left(-\frac{z}{\lambda\alpha}\right)\right) = \lambda\left[-\alpha\, g\left(g\left(-\frac{z}{\lambda}\right)\cdot\frac{1}{\alpha}\right)\right]$$

Same functional equation!

"Fix λ" by demanding $g(0) = 1$.

Approximate $g(z) \approx 1 + g_2 z^2 + \cdots$

Solve for g_2 & α self-consistently:

$$g(z) = -\alpha\, g\left(g\left(-\frac{z}{\alpha}\right)\right)$$

$$1 + g_2 z^2 = -\alpha\left(1 + g_2\left(1 + g_2\left(-\frac{z}{\alpha}\right)^2\right)^2\right) + \cdots$$

$$= -\alpha(1+g_2) - \alpha\, g_2 \cdot \frac{2g_2}{\alpha^2} z^2 + \cdots$$

$1 = -\alpha(1+g_2)$ $\qquad$ $g_2 = -\alpha\,\frac{2g_2^2}{\alpha^2}$ or $g_2 = -\frac{\alpha}{2}$.

$1 = -\alpha + \frac{\alpha^2}{2}$. Solve: $\alpha = 1 \pm \sqrt{3}$

Physical solution needed $\alpha > 1$, so $\alpha \approx 1 + \sqrt{3} \approx 2.73$.

Use higher order polynomial approx for g? $\quad \alpha \to 2.502\ldots$