PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 4
Time-translation and boost symmetry

September 4

Continuous symmetry leaves EOM invariant:

$$x_i \to x_i + \varepsilon X_i + \cdots \qquad t \to t + \varepsilon T + \cdots$$

generator

Lagrangian is "symmetric": repeated index $= \sum_i$

$$\dot{\Phi}(x_i, t) + \dot{T}L + T\frac{\partial L}{\partial t} + X_i \frac{\partial L}{\partial x_i} + (\dot{X}_i - \dot{T}\dot{x}_i)\frac{\partial L}{\partial \dot{x}_i} = 0.$$

Noether's Thm: continuous sym $\Rightarrow$ conservation law:

$$Q = T\left(L - \dot{x}_i \frac{\partial L}{\partial \dot{x}_i}\right) + X_i \frac{\partial L}{\partial \dot{x}_i} + \Phi,$$

$\dot{Q} = 0$ on phys traj.

Example 1: Time-translation symmetry

$$x_i \to x_i \quad \& \quad t \to t + \text{const.}$$

generated by $(X_i, T) = (0, 1)$

Almost always postulated.

Claim: can assume $\frac{\partial L}{\partial t} = 0$ (i.e. L, not S, invariant).

Proof: $\dot{\Phi} + \frac{\partial L}{\partial t} = 0.$ Write: $\Phi = \frac{\partial \Psi(x,t)}{\partial t}.$

$$\frac{d}{dt}\Phi = \frac{\partial \Phi}{\partial t} + \dot{x}_i \frac{\partial \Phi}{\partial x_i} = \frac{\partial^2 \Psi}{\partial t^2} + \dot{x}_i \frac{\partial^2 \Psi}{\partial x_i \partial t} = \frac{\partial}{\partial t}\dot{\Psi}$$

So : $0 = \frac{\partial}{\partial t}\Big[\underbrace{L + \frac{d\Psi}{dt}}\Big]$

freely shift L by total der. w/o changing EOM

So $\tilde{L} = L + \frac{d}{dt}\Psi$ is a valid Lagrangian : $\frac{\partial \tilde{L}}{\partial t} = 0.$

Noether Thm : conserved $-E = L - \dot{x}_i \frac{\partial L}{\partial \dot{x}_i}$

energy (define)

E.g. central force Lagrangian: $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$

$\hookrightarrow$ $E = \underbrace{\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)}_{= \frac{1}{2}mv^2 \text{ (kinetic)}} + \underbrace{V(r)}_{\text{potential}}$

Example 2: Galilean boost symmetry. Configuration space $\mathbb{R}$.

$\tilde{x} = x + vt$ & $\tilde{t} = t$: $(X, T) = (t, 0)$

constant

AND time-translation. : $(X, T) = (0, 1)$

Recall: composing multiple sym trans $\rightarrow$ new? sym trans.

So: Step 1 : $\tilde{x} = x + vt$ (const) $\quad \tilde{t} = t$

Step 2 : $\tilde{\tilde{x}} = x + \underbrace{v(t}_{\text{boost}} + \underbrace{t_0)}_{\text{space translation}} \quad \tilde{\tilde{t}} = t + \underbrace{t_0}_{\text{time-trans.}}$ (HW1 Prob 4)

Generators do not commute. (non-Abelian symmetry group)

What's most general invariant L?

TT sym: $L(x, \dot{x})$

space trans: $L(\dot{x})$

Boost: $x \to x + vt$, $\dot{x} \to \dot{x} + v$.

If we want L to be invariant: $L(\ddot{x})$ only...

Now consider Φ!

$$\hookrightarrow \quad \dot{\Phi} + t\cancel{\frac{\partial L}{\partial x}} + \frac{\partial L}{\partial \dot{x}} = 0 \qquad (X,T) = (t,0)$$

$\Phi(x,t)$ only

$$\downarrow$$

$$\underbrace{\cancel{\frac{\partial \Phi}{\partial t}}}_{\text{time-trans}} + \underbrace{\dot{x}\frac{\partial \Phi}{\partial x}}_{\dot{x}\,\boxed{func(x)}} + \underbrace{\frac{\partial L}{\partial \dot{x}}}_{func(\dot{x}),\ \text{since } \frac{\partial L}{\partial x} = 0} = 0$$

const.

$$\Phi = -mx \;\to\; \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \text{or} \quad L = \frac{1}{2}m\dot{x}^2$$

$$\frac{\delta S}{\delta x} = \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -\frac{d}{dt}(m\dot{x}) = -m\ddot{x} = 0$$

(force-free Newton's 2nd Law)

Hence $L = \frac{1}{2}m\dot{x}^2 + F(\ddot{x}, \ldots)$.

Effective theory valid on long time scales $\Delta t \gg \tau$...

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}c\ddot{x}^2 - \cancel{\frac{1}{3}c\dddot{x}^3} + \cdots$$

[Taylor expand F!)

Euler-Lagrange equation:

$$\frac{\delta S}{\delta x} = \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial^2 L}{\partial \ddot{x}} = 0 = -m\ddot{x} - c\,x^{(4)}$$

If we include $c \neq 0$, most general solution is

$$x(t) = \underbrace{a + bt}_{\text{slow dynamics captured by eff. th.}} + \underbrace{c_1\cos(\Omega t) + c_2\sin(\Omega t)}_{\text{"internal dynamics" if } \Omega \sim 1/\tau}, \qquad \Omega = \sqrt{\frac{m}{c}}$$

integration const.

↳ also sol'ns to $m\ddot{x} = 0$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \tau^2 \ddot{x}^2 + \dots$$

"microscopic" time

Alternative: Taylor expanding L in $\tau \frac{d}{dt}$

Effective theory: keeping leading order τ^0.

Noether's Thm: $\left(L = \frac{1}{2} m \dot{x}^2\right)$

↳ trans (x): $p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$ const.

boost: $Q = \Phi + t \frac{\partial L}{\partial \dot{x}} = -mx + pt$

Plug in $t=0$:

$$Q = -m x(0)$$

so: $x(t) = x(0) + \frac{p}{m} t$

Most general "phys traj" w/o Euler-Lagrange!