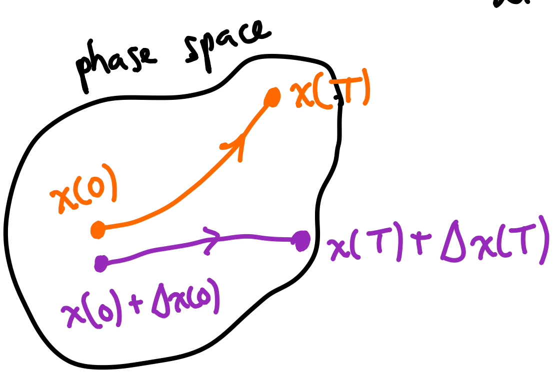
PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 40
The butterfly effect

December 4

<u>Butterfly effect:</u> exponential sensitivity of chaotic dynamics to initial conditions.



$$\Delta x(T) \sim \Delta x(0) \cdot e^{\lambda T}$$

Lyapunov exponent $\underline{\lambda > 0}$.

Write as

$$\frac{\partial x(T)}{\partial x(0)} \sim e^{\lambda T}$$

Is butterfly effect a diagnostic for chaos?

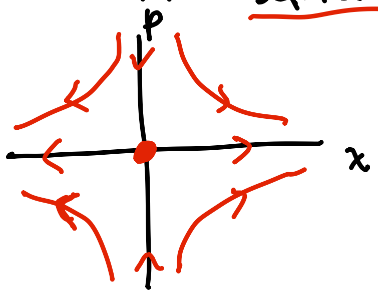
▷ No (literally). Hamiltonian solvable system w/ exponential diverging trajectories...

Take phase space $\mathbb{R}^2$, $\{x, p\} = 1$.
Take $H = xp$.

$$\dot{x} = \frac{\partial H}{\partial p} = x \quad \text{so} \quad x(T) = x(0)e^{T} \quad \text{so} \quad \frac{\partial x(T)}{\partial x(0)} = e^{T}$$

Have "Lyapunov exponent" $\lambda = 1$.

Problem: saddle point.



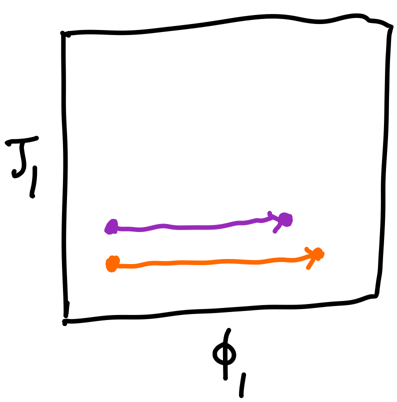
Any initial condition w/ $x(0) \neq 0$ exhibits fake "butterfly effect"...

lesson: do not count saddle pts when looking for Lyapunov...

$\triangleright$ Yes! Butterfly effect good, ignoring saddles...

... integrable Hamiltonian system has patches w/ action-angle variables...

$H(J_A)$: $\quad \dot{\phi}_A = \dfrac{\partial H}{\partial J_A} = \text{const.} \qquad \& \qquad \dot{J}_A = -\dfrac{\partial H}{\partial \phi_A} = 0.$



$$\text{"}\Delta x(T)\text{"} \sim \sqrt{\Delta J_1(T)^2 + \Delta\phi_1(T)^2}$$

$$\sim \sqrt{\Delta J_1(0)^2 + \left(\Delta\phi_1(0) + T\cdot\Delta\omega_1\right)^2}$$

$$\sim T\cdot\Delta\omega_1 \qquad \hookrightarrow \omega_1 = \frac{\partial H}{\partial J_1}$$

not exponential growth, unlike butterfly effect.

Generic initial conditions avoid saddle, so generic ICs won't have butterfly effect...

lec 43: chaos in Hamiltonian systems $\rightsquigarrow$ "proliferation of saddles..."

Return to logistic map: $\quad x_{n+1} = r x_n (1-x_n) \qquad 0 \le x_n \le 1$

$$\underbrace{0 \le r \le 4}$$

$x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$

vs.

$x_0 + \delta x_0 \rightarrow x_1 + \delta x_1 \rightarrow x_2 + \delta x_2 \rightarrow \cdots$

Choose $r = 3.6$

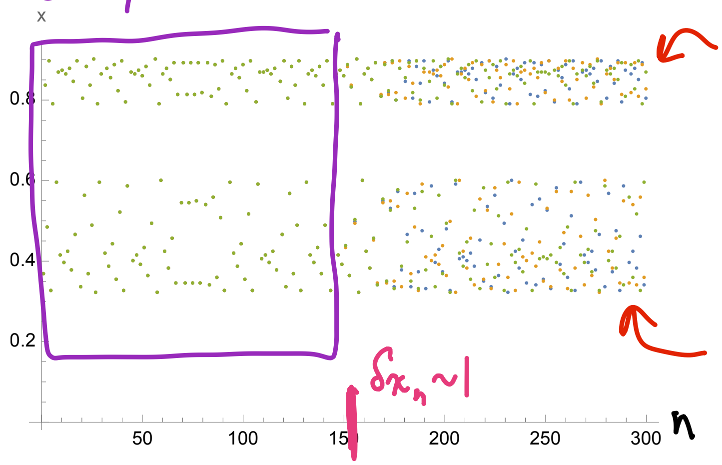
chaos

Generic IC, do expect $\delta x_n \sim O(1)$ as $n \to \infty$. Is it exponentially fast?

Numerical exp: 3 runs w/ different ICs:

$x_0 = 0.37$ $\quad$ $x_0 = 0.37 + 10^{-15}$ $\quad$ $x_0 = 0.37 - 10^{-15}$

early times: cannot distinguish 3 trajectories!



butterfly effect?

$\delta x_{300} \sim O(1)$

Estimate Lyapunov exponent:

$$\delta x_n \sim \underbrace{\delta x_0}_{10^{-15}} \cdot e^{\lambda n} \longrightarrow \text{Plug in } n = 150:$$

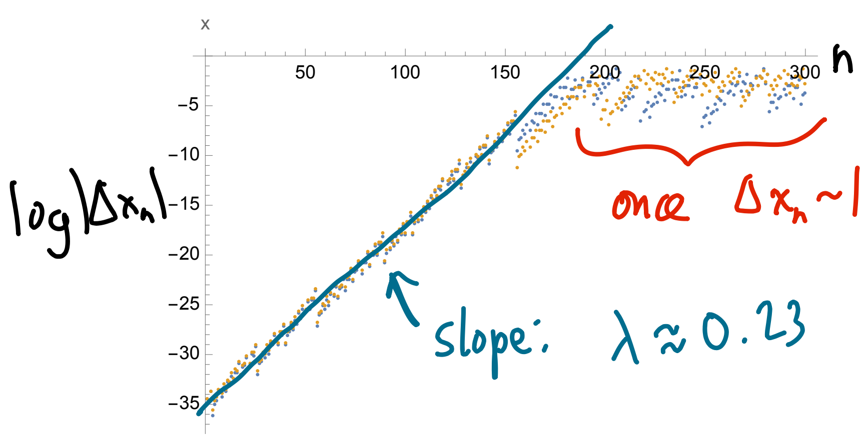
$1 \sim 10^{-15} e^{\lambda \cdot 150}$

$15 \log 10 \sim 150\lambda$

or $\lambda \sim 0.23$

Claim: Lyapunov exponent λ (≈ 0.23 here) independent of x_0.

Numerically confirm exponential growth directly:



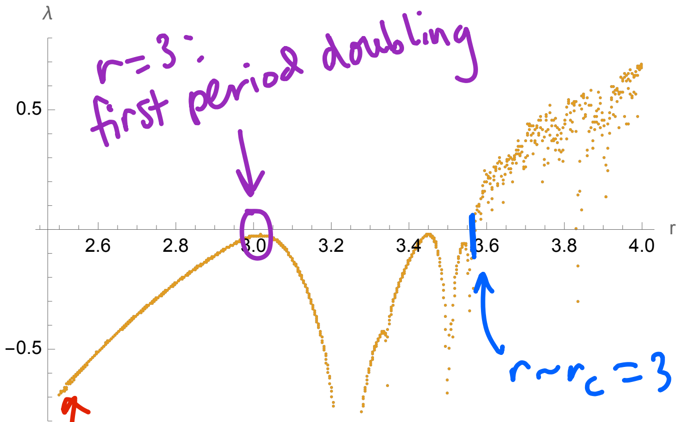
once $\Delta x_n \sim 1$, butterfly growth stops...

so: $\log|\Delta x_n| \approx \log|\Delta x_0| + \lambda n$: indeed, $\underbrace{\text{butterfly effect.}}_{\text{b/c we saw exponential growth!}}$

Claim: for logistic map, as a function of r, $\lambda > 0$ heralds chaos.

Numerics: for multiple ICs, extract slope $\log|\Delta x_n| \approx \log|\Delta x_0| + \lambda n \ldots$

$r=3$: first period doubling



one stable fixed pt.

$r \sim r_c = 3.57\ldots$ onset of chaos: $\lambda > 0$! (butterfly effect).

As we move through period doubling, $\lambda \to 0$ from below... and goes negative after $r=3$ $\rightarrow$ $\underbrace{\text{Stable period-2 cycle}}$

λ captures how quickly we approach period-2 cycle.