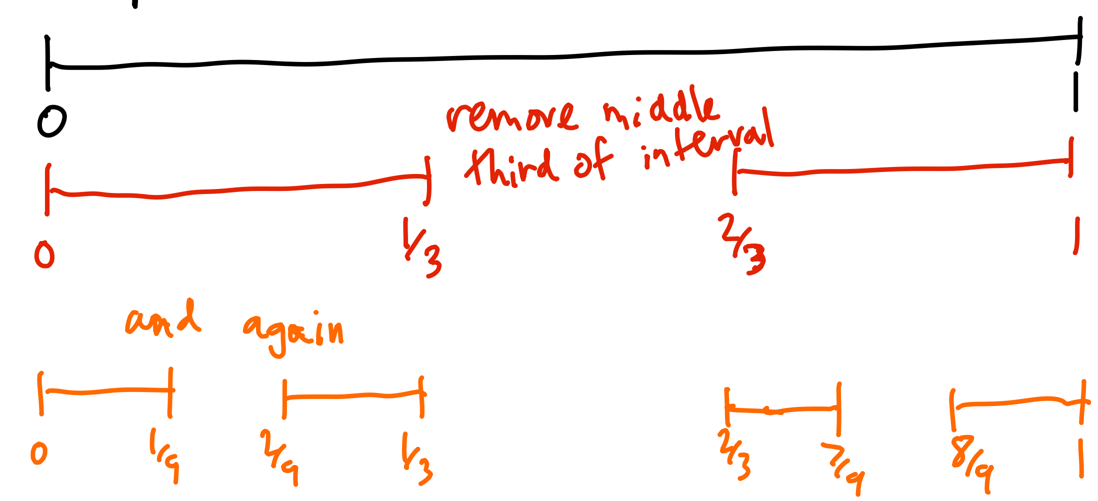
PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 41
Fractals

December 6

Fractal = set w/ "structure" at arbitrarily small scale.

Example 1: Cantor set.



$S_0 = [0,1]$

$S_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

S_2

Crudely: Cantor set $S =$ "$\lim_{n\to\infty} S_n$".

- S has ∞ points: $0, \frac{1}{3}, \frac{1}{9}, \ldots \in S$. uncountable
- "S has no volume": $\mathrm{Vol}(S_0) = \int_0^1 dx = \int_{S_0} dx = 1$

$$\mathrm{Vol}(S_1) = \int_{S_1} dx = \frac{2}{3} = \int_0^{1/3} dx + \int_{2/3}^1 dx$$

Iterate: $\mathrm{Vol}(S_n) = \left(\frac{2}{3}\right)^n$ so $\mathrm{Vol}(S) = 0$.

Cantor: both uncountable, yet fraction of number line $= 0$.

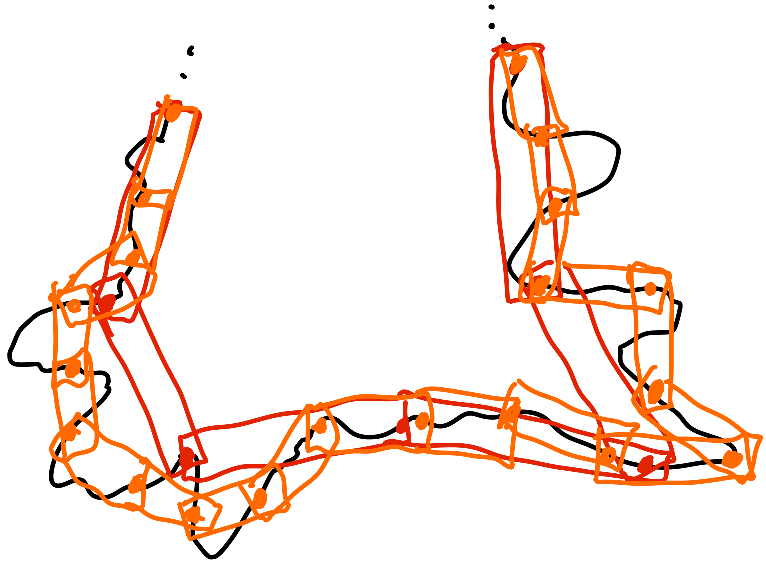
S = fractal (more later). For now: S has structure (holes) at all scales

---

Example 2: coastline length (Mandelbrot)

600 km ← (L ~ 100 km) b/c I used 6 rulers

760 km ← (L ~ 40 km)



How can my 2 lengths disagree??

Calculus: ruler length $\varepsilon \to 0$... length $\sim \int d\ell$

does this limit exist??

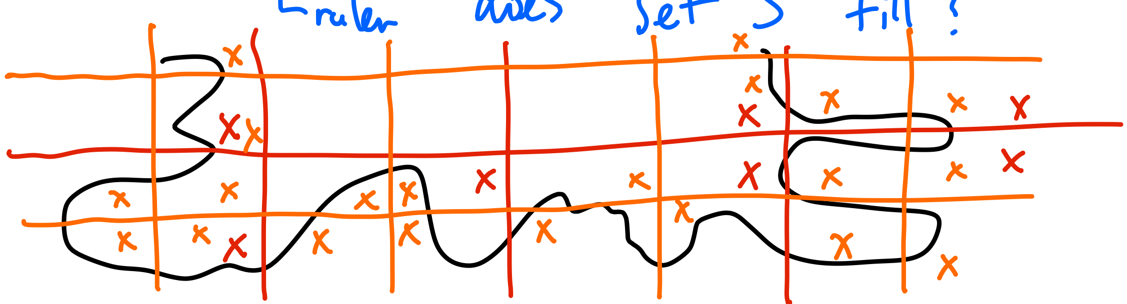
Better perspective: coast length L_{coast} depends on L_{ruler}:

$$L_{coast} \sim \frac{1}{L_{ruler}^{d}} \qquad \boxed{d \approx 1.2.}$$

non-integer dimension characterizes our fractal...

For fractals, many notions of dimension...

For today: box dimension. "How many boxes of side length L_{ruler} does set S fill?"



Scale L_0

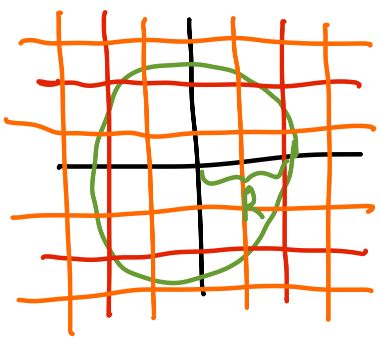
Scale $L_1 = \frac{1}{2} L_0$

let N_n = # of boxes at scale n that contain curve S.

$N_0 = 7$ $\quad N_1 = 21 \quad N_2 = \cdots \quad N_3 = \cdots$

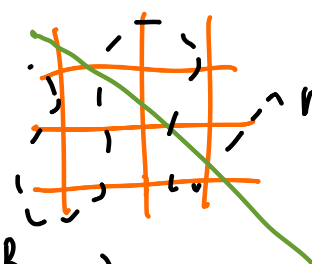
Define box dimension $d = \lim\limits_{\substack{n\to\infty \\ (L_n \to 0)}} \dfrac{\log N_n}{\log\left(\frac{1}{L_n}\right)}$, i.e. $N_n \sim \dfrac{1}{L_n^d}$

Example 3: embed a circle (1-dim...) into $\mathbb{R}^2$



Estimate: $N_n \simeq \dfrac{2\pi R}{L_n} \cdot O(1)$

because:



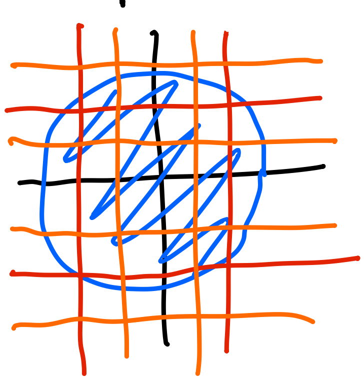
needed for fractal

circle

Box dimension $d = \lim\limits_{L_n \to 0} \dfrac{\log\left(\frac{2\pi R}{L_n} \cdot c\right)}{\log\left(\frac{1}{L_n}\right)} = \lim\limits_{L_n \to 0} \dfrac{\log(2\pi R c) + \log\left(\frac{1}{L_n}\right)}{\log\left(\frac{1}{L_n}\right)}$

$= 1.$ Circle is 1-dim! !!

Example 4: fill in the circle.



$N_n \geq \dfrac{\pi R^2}{L_n^2}$ (also: L_n small enough $N_n \leq \dfrac{\pi (2R_n)^2}{L_n^2} \ldots$)

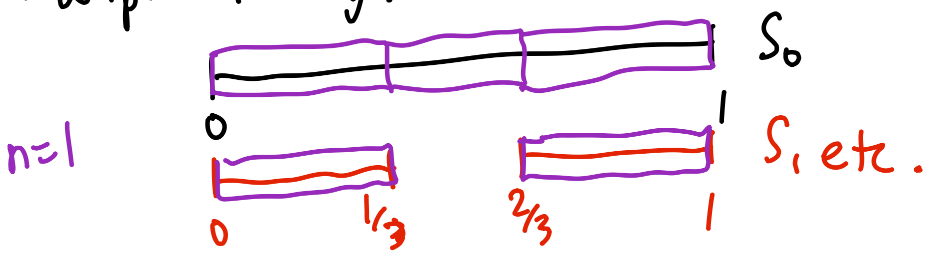
Box dimension:

$d = \lim\limits_{L_n \to 0} \dfrac{\log\left(\frac{\pi R^2}{L_n^2}\right)}{\log \frac{1}{L_n}} = \lim\limits_{L_n \to 0} \dfrac{2 \log \frac{1}{L_n} + \log(\pi R^2)}{\log \frac{1}{L_n}}$

$= 2.$

Box dimension gets dimension of "ordinary shapes" right, regardless of what dimension we embedded into.

Example 1 again: Cantor set.



Choose boxes of size $L_n = 3^{-n}$. Count box if interior lies in Cantor set.

Cover S_n in 2^n boxes of size L_n.

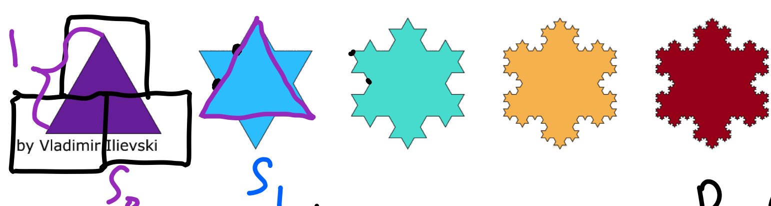
Since $S \subset S_n$, also cover S in 2^n boxes.

So box dimension $d = \lim_{n\to\infty} \dfrac{\log N_n}{\log \frac{1}{L_n}} = \lim_{n\to\infty} \dfrac{n \log 2}{n \log 3} = \dfrac{\log 2}{\log 3} \approx 0.63$.

So Cantor set has a non-integer box dimension.
$\Rightarrow$ structure at all scales! $\Rightarrow$ fractal!

Example 5: Koch snowflake

Box dimension of boundary?



Box length $L_n = 3^{-n}$.

use rescaling $\frac{1}{3}$ factor b/c each line segment drops by factor $1/3$ in length.

$N_n \sim \dfrac{P_n}{L_n} \cdot O(1)$.

perimeter at iteration n

$P_n = \frac{4}{3} P_{n-1}$ or $P_n = 3 \cdot \left(\frac{4}{3}\right)^{n-1}$

Box dimension: $d = \lim_{n\to\infty} \dfrac{\log(P_n/L_n)}{\log(\frac{1}{L_n})} = \lim_{n\to\infty} \dfrac{n\log\frac{4}{3} + n\log 3}{n\log 3} = \dfrac{\log 4}{\log 3} \approx 1.26$