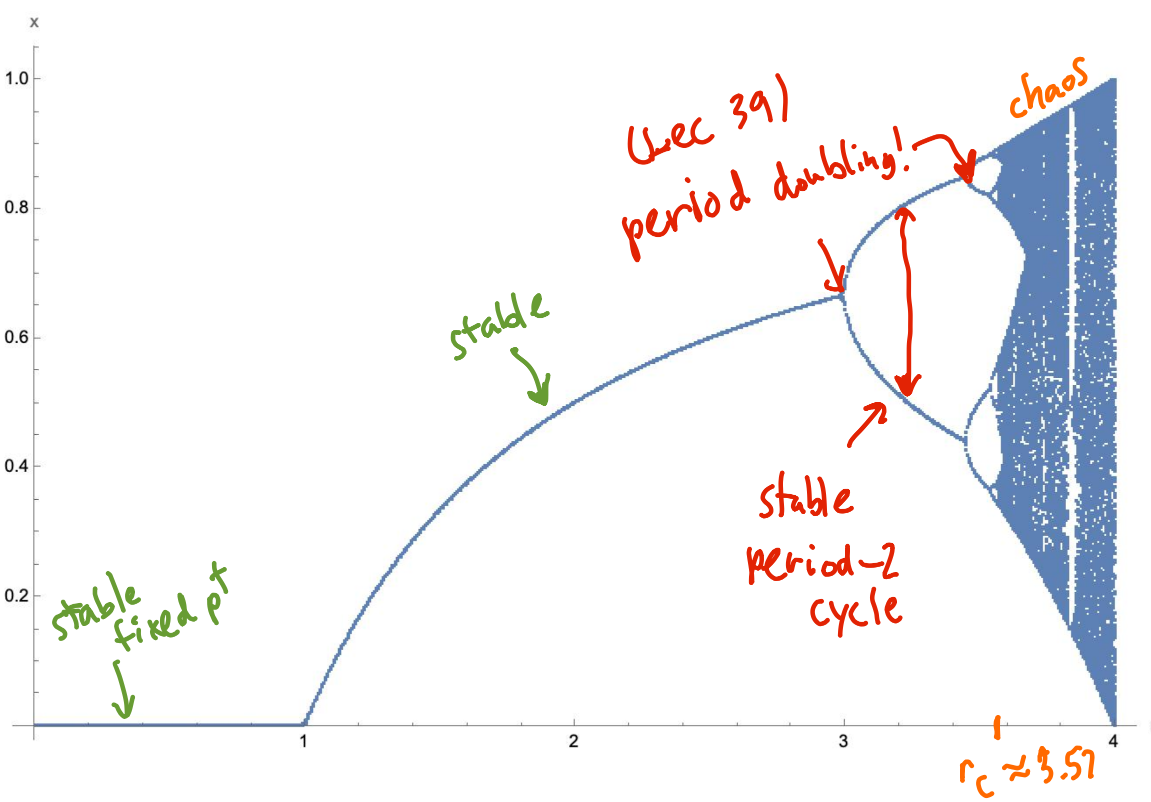
PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 42
Fractal attractor in the logistic map

December 9

Logistic map: $x_{n+1} = r x_n (1 - x_n)$ $\quad 0 \le x_n \le 1$, $\quad 0 \le r \le 4$

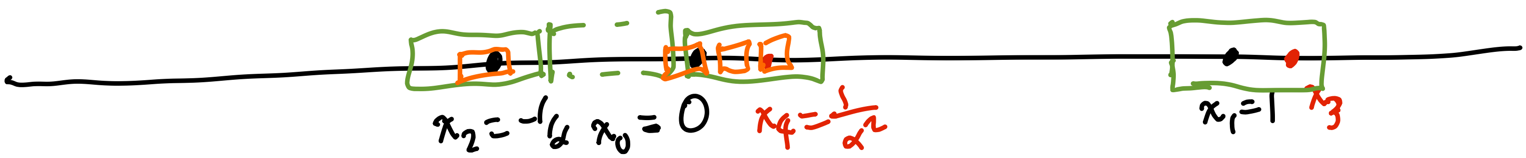
Attractor: at each r, collect set of points x_n as $n \to \infty$.



Claim: at $r = r_c$, attractor of logistic map is fractal: exhibits structure at all scales

Lec 41: characterize fractal via box dimension:
if it takes $N(\varepsilon)$ boxes of size ε to enclose set S,
S has box dimension $d = \lim_{\varepsilon \to 0} \dfrac{\log N(\varepsilon)}{\log \frac{1}{\varepsilon}}$ (i.e. $N(\varepsilon) \sim \frac{1}{\varepsilon^d}$)

Use period doubling/RG (Lec 39) to estimate d.
↳ the iterated map $f^{[2^k]}(x)$ (when rescaled) approaches $g(x)$ defined by $g(g(x)) = -\frac{1}{\alpha} g(-\alpha x)$. $\alpha \approx 2.5$, $g(0) = 1$.
if $x_n = g(x_{n-1})$



$x_0 = 1$, $x_2 = -1/\alpha$, $x_4 = \frac{1}{\alpha^2}$, ...

Claim: self-similarity arises due to $g(g(x)) = \dots$

...so to estimate box dimension:

$L_n = \frac{1}{\alpha^2} L_{n-1}$ ⟶ right: ~ 1
left: ~ 3. morally, α

$\varepsilon \sim \alpha^{2n}$

So $d = \lim_{n \to \infty} \dfrac{\log((\alpha+1)^n)}{\log \frac{1}{\alpha^{2n}}} \sim \dfrac{\log(\alpha+1)}{2 \log \alpha} \approx 0.68$

Qualitative similarity w/ Cantor set example (Lec 41).

Usually, people report dimension of $r = r_c$ attractor as $d \sim 0.5 - 0.54$. Our cartoon isn't very good.

Issue: no good prescription for where boxes go...
box dimension can be ambiguous...

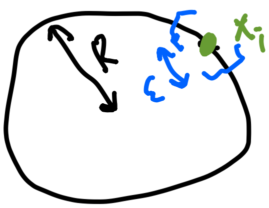
More savvy approach for numerics: Correlation dimension d_2

- Run logistic map: $x_0 \to x_1 \to x_2 \to \cdots \to \underbrace{\cdots \to x_N}_{}$ (generic)

  Keep only "late time" data, sample attractor?

- How many pairs (x_i, x_j) from remaining set obey $|x_i - x_j| < \varepsilon$. Call this $P(\varepsilon)$

  ↳ If set was uniformly sampled in $d=1$ ...



→ estimate: $P(\varepsilon) \sim N \underbrace{\left(\frac{\varepsilon}{R}\right)^d N}_{\#\text{ of } x_j \text{ w/in } \varepsilon} \cdot \frac{1}{2}$

pick x_i

$|x_i - x_j| = |x_j - x_i|$

or $P(\varepsilon) \sim \frac{N^2}{2} \cdot \left(\frac{\varepsilon}{R}\right)^d \sim \varepsilon^d$

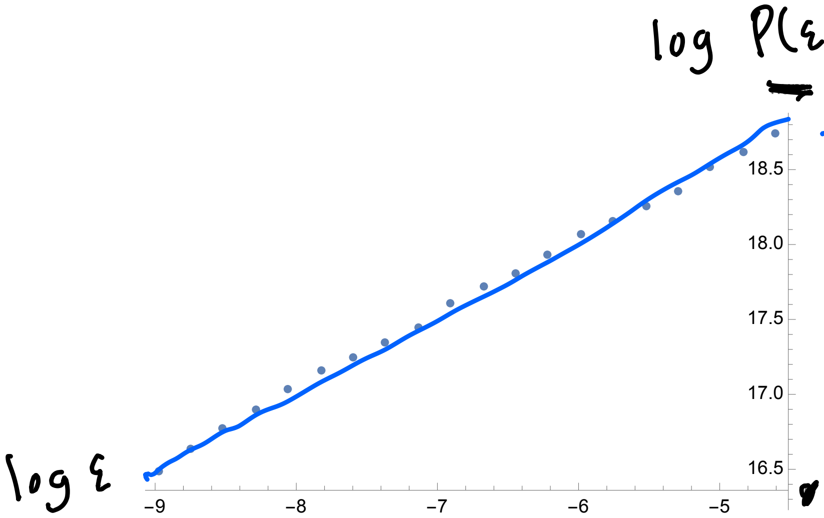
To calculate our correlation dimension d_2:

$$d_2 = \lim_{\varepsilon \to 0} \left(\lim_{N\to\infty} \frac{\log P(\varepsilon) - \log N^2}{\log(\varepsilon)} \right) = \lim \frac{\log \cancel{\frac{N^2}{2}} + d \log \frac{\varepsilon}{R} - \cancel{\log N^2}}{\log \varepsilon}$$

Define d_2 as a new kind of dimension. Can be different from other notions of dimension.

d_2 is very easy to estimate numerically!

Numerical experiment: calculate d_2 for logistic map attractor at $r = 3.57$ at $N = 5 \times 10^4$



→ Mathematica fit: Slope $= d_2 \approx 0.50$, close to what we claimed before.

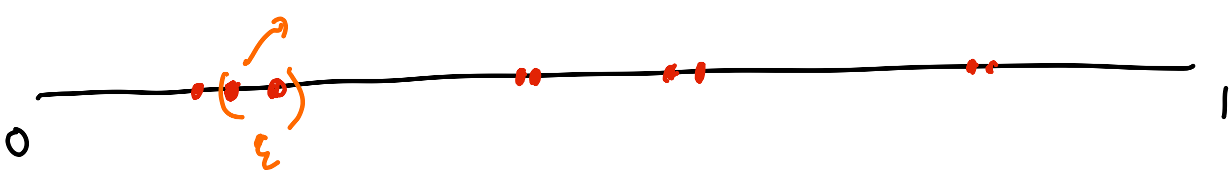
$\varepsilon \sim 10^{-4}$?

$\varepsilon > \frac{1}{N}$.

$\varepsilon \sim 10^{-2}$

Like box dimension, non-integer $d_2 \Rightarrow$ structure at all scales.

# of neighbors $\sim \varepsilon^d \sim \sqrt{\varepsilon}$



So attractor is very closely packed. NOT random.