PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 43
KAM Theorem

December 11

Recall: integrable Hamiltonian has action-angle variables (ϕ_A, J_A)

$$H = H_0(J_A)$$

$$\dot{J}_A = -\frac{\partial H}{\partial \phi_A} = 0 \quad \text{and} \quad \dot{\phi}_A = \frac{\partial H}{\partial J_A} = \omega_A = \text{const.}$$

lec 33: perturbation theory for integrable system:

$$H = H_0(J_A) + \varepsilon H_1(\phi_A, J_A) \quad \text{w/} \quad H_1 = \sum_{\vec{m}} e^{i\vec{m}\cdot\vec{\phi}} h_{\vec{m}}.$$

Goal: new AA variables via $\underbrace{\text{Type 2 CT}}$ (or "Lie" method")

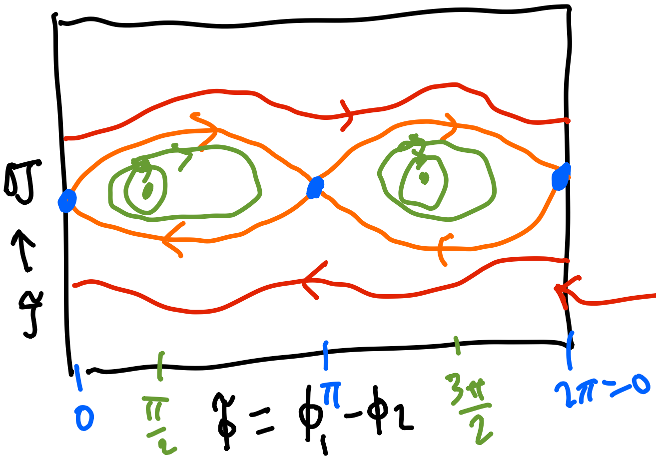
$$S(\vec{\phi}_0, \vec{J}) = \vec{\phi}_0 \cdot \vec{J} + \varepsilon \sum_{\vec{m}} e^{i\vec{m}\cdot\vec{\phi}} \frac{h_{\vec{m}}(\vec{J})}{\underbrace{i\vec{m}\cdot\vec{\omega}}}$$

Perturbation theory breaks down if $\vec{m}\cdot\vec{\omega} = 0$ (commensurate freq.). Even as $\varepsilon \to 0$.

If "all" $h_{\vec{m}} \neq 0$, expect $\vec{m}\cdot\vec{\omega}(\vec{J}) = 0$ occurs ∞ many points.

↳ need to estimate how much of phase space has PT failure, at some fixed $\varepsilon > 0$.

lec 33:



Suppose a resonance at $m=2$:

$$H \sim \varepsilon \cos(m\tilde\phi) + \tilde J^2$$

old AA continue to exist.

We always have
- m stable pts $\rightarrow$ "islands of integrability"
- m unstable pts $\rightarrow$ chaos

Consider discrete "Hamiltonian/symplectic" map (e.g. kicked rotor/ lec 36)

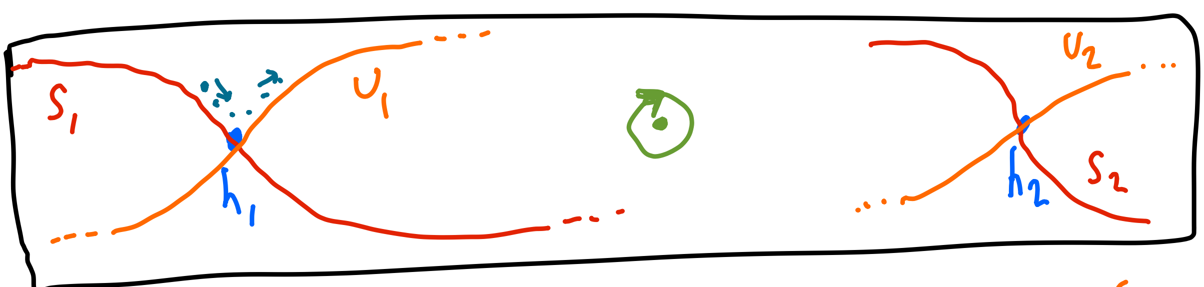
$$\begin{pmatrix} J_{n+1} \\ \phi_{n+1} \end{pmatrix} = \begin{pmatrix} J_n + \varepsilon \sin\phi_n \\ \phi_n + J_n + \varepsilon \sin\phi_n \end{pmatrix} = Z \cdot \begin{pmatrix} J_n \\ \phi_n \end{pmatrix}$$

<u>not</u> matrix

Take n=1 AA pair

Important: Z^{-1} exists. Hamiltonian mechanics is reversible.

Look at map near unstable (hyperbolic) fixed point h_i "$\begin{pmatrix} J \\ \phi \end{pmatrix}$"
$[Z \cdot h_i = h_i]$



U_1 = "unstable manifold": "points driven away from h_1" (?)

$$\lim_{n\to\infty} Z^{-n} \cdot x = h_1$$

S_1 = "stable manifold": set of pts x where

$$\lim_{n\to\infty} Z^{n} \cdot x = h_1$$

only intersection is h_1

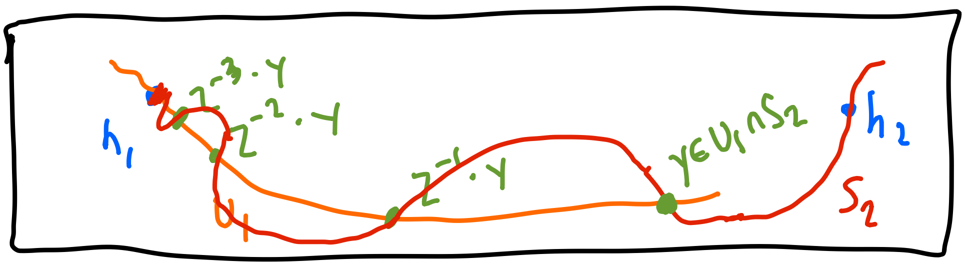
Claim: S_1 & S_2 don't intersect.

Proof: Suppose $x \in S_1 \cap S_2$ ($x \in S_1$ & $x \in S_2$). Then

$\lim_{n\to\infty} Z^n \cdot x = h_1$ $\lim_{n\to\infty} Z^n \cdot x = h_2$

which is a contradiction. So $S_1 \cap S_2$ = empty.

Also, U_1 & U_2 don't intersect.



Claim: U_1 & S_2 can intersect.

Claim: if y exists, $U_1 \cap S_2$ has ∞ points.

Proof: If $y \in U_1 \cap S_2$, so is $Z^\ell \cdot y$ for any ℓ.

$$\lim_{n\to\infty} Z^n \cdot (Z^\ell \cdot y) = \lim_{n\to\infty} Z^{n+\ell} \cdot y = \lim_{n\to\infty} Z^n \cdot y = h_2 \quad \in S_2$$

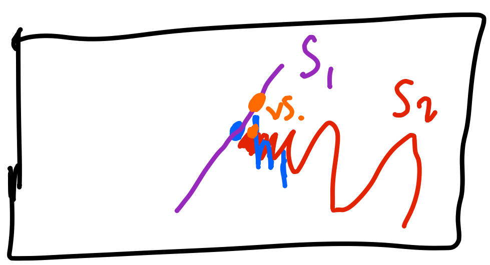
$$\lim_{n\to-\infty} \text{'' } \ldots \text{ ''} = h_1 \quad \in U_1$$

Claim: chaos arises out of complex intersection btwn S_2 & U_1

Intuitive: S_2 very close to h_1

BUT S_1 passes through h_1, S_1 don't intersect S_2.

Zoom in on h_1:



chaos? very different trajectories for nearby pts...

...and S_n are complex / fractal?

Claim: in general, chaos exists somewhere in phase space at any $\varepsilon \neq 0$.

KAM Theorem: If ε small enough, $H = \underset{\uparrow \text{integrable}}{H_0} + \varepsilon H_1$

then system is integrable in "finite fraction of phase space".

Idea: Perturbation $\rightsquigarrow$ construct new AA vars thru Type 2 CT, generating func:

$$S = \vec{\phi}_0 \cdot \vec{J} + \varepsilon \sum_{\vec{m}} e^{i\vec{m}\cdot\vec{\phi}_0} \boxed{\frac{h_{\vec{m}}(\vec{J})}{i\vec{m}\cdot\vec{\omega}(\vec{J})}} \rightarrow \text{if small, maybe PT converges?}$$

Resonances occur when $\vec{m}\cdot\vec{\omega} \to 0$ ... if infinitely many $h_m \neq 0$, expect ∞ resonances...

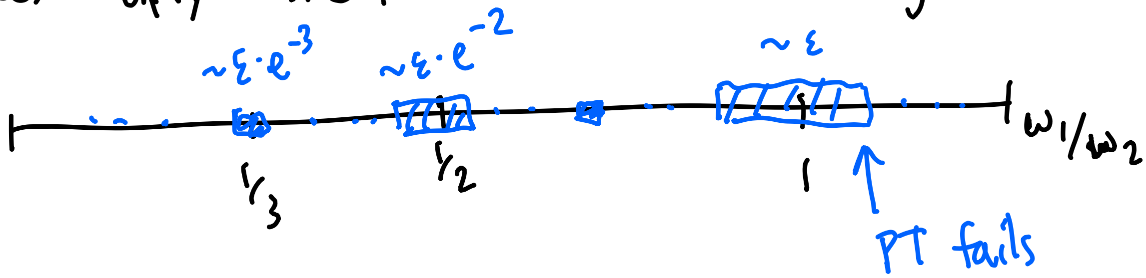
We want $\dfrac{\vec{m}\cdot\vec{\omega}(\vec{J})}{h_m(\vec{J})} \gg \varepsilon$ for PT to converge (?)

Claim: many physical systems... $h_m \sim e^{-\alpha|\vec{m}|}$.

Diaphantine condition: most $\vec{\omega}$ (incommensurate) obey $|\vec{m}\cdot\vec{\omega}| \gtrsim \gamma \cdot |\vec{m}|^{-\kappa}$

γ, κ $O(1)$

Since $e^{\alpha|\vec{m}|}\gamma|\vec{m}|^{-\kappa} \to \infty$ as $|\vec{m}| \to \infty$, so "high order resonances" only break PT in small region.



Cantor-like set of phase space where PT fails...
failure region is "dense", but PT works at typical points.

How to show PT converges? Techniques from Lec 34.

$$H \to H_0 + \varepsilon H_1 \rightsquigarrow H_0 + \varepsilon^2 \tilde{H}_2 \longrightarrow H_0 + \varepsilon^4 \tilde{H}_4 \to \cdots$$