PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 6
Relativistic particles

September 9

Goal: today: action for relativistic particle of mass m
next: charged w/ EM fields

Step 1: identify symmetries:

① translation:

const.

$$t \to t + \varepsilon_t$$
$$x \to x + \varepsilon_x$$
$$y \to y + \varepsilon_y$$
$$z \to z + \varepsilon_z$$

4 independent translation generators

3 total

② rotation:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

rotate around x, y, z-axis

3 total

③ boost:

$$\begin{pmatrix} ct \\ x \end{pmatrix} \to \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

move in the x, y, z -direction

where $\beta = \frac{v}{c}$ and $\gamma = (1-\beta^2)^{-1/2}$.

How many continuous symmetries? $4 + 3 + 3 = 10$.

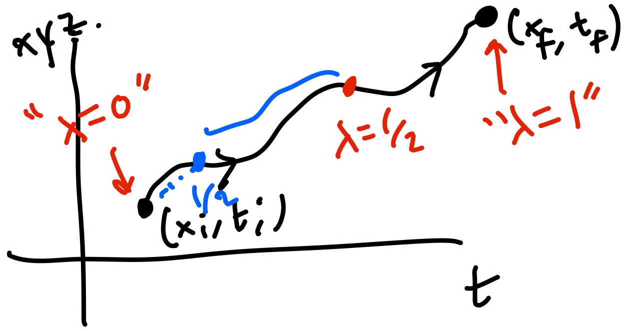
completely fix S!

Note: t & x on (almost) equal footing

↳ but $S = \int dt \, L(x, \dot{x}, \ldots, t)$

$\dot{x} = \frac{dx}{dt}$

Idea: introduce "fake time λ":



Arbitrary choice of λ so long as if $\lambda_1 < \lambda_2$, $t(\lambda_1) < t(\lambda_2)$

"time reparamaterization"

λ is unphysical: $\lambda \longrightarrow f(\lambda)$ so long as f is invertible

Goal: $S[t(\lambda), x(\lambda), y(\lambda), z(\lambda)] = \int d\lambda \, L\Big(\underbrace{x^\mu, \frac{dx^\mu}{d\lambda}, \ldots}\Big)$

Notation $x^\mu = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$

Work in units where $c = 1$.

Step 2: Identify invariant building blocks for L.
(ignore reparamaterization)

① Translations: $x^\mu \to x^\mu + \varepsilon^\mu$ $(\varepsilon_t, \varepsilon_x, \varepsilon_y, \varepsilon_z)$

↓ this implies

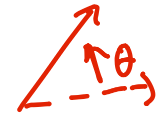
$\frac{dx^\mu}{d\lambda}$ is invariant ... $\frac{\partial L}{\partial x^\mu} = 0.$ (lec 2&3)

② & ③ Rotation & boost = Lorentz group

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

Under just z-rotation: t, z are invariant
invariant(x, y) = $x^2 + y^2$



Add x/y-rotation: $\underbrace{t, \; x^2+y^2+z^2}_{\text{combined invariant under boost}}$ invariant

$$\underbrace{-\Delta t^2 + (\Delta x^2 + \Delta y^2 + \Delta z^2)}_{} = -\Delta\tau^2$$

← "proper time"

Write as $\underbrace{\eta_{\mu\nu}\Delta x^\mu \Delta x^\nu}_{}$

repeat index: $\sum_{\mu\nu}$ (for relativistic: one upper/one lower)

where $\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

In relativity, "use $\eta_{\mu\nu}$ to lower indices":

$x^\mu = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$, then $\eta_{\mu\nu}x^\nu = x_\mu = \begin{pmatrix} -t \\ x \\ y \\ z \end{pmatrix}$

So $\eta_{\mu\nu}x^\mu x^\nu = x_\mu x^\mu$. Analogue of vector's length.

Alternate picture: start w/ postulate that $x_\mu x^\mu$ is invariant under Lorentz

Useful: objects traveling @ $v = c$ have $\Delta x_\mu \Delta x^\mu = 0$ in all frames, all observers agree on speed of light

Look for infinitesimal coordinate change: $x^\mu \to \Lambda^\mu{}_\nu x^\nu$
↑ const. matrix

$$\Lambda^\mu{}_\nu = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\delta^\mu_\nu} + \varepsilon^\mu{}_\nu$$

w/ $\eta_{\mu\nu}x^\mu x^\nu = \eta_{\mu\nu}(\Lambda^\mu{}_\alpha x^\alpha)(\Lambda^\nu{}_\beta x^\beta)$

Taylor expand to first order in ε:

$$\eta_{\alpha\beta} = \eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\mu\nu}(\delta^\mu_\alpha + \varepsilon^\mu{}_\alpha)(\delta^\nu_\beta + \varepsilon^\nu{}_\beta)$$

$\rightarrow$ first order: $\quad 0 = \eta_{\mu\beta}\varepsilon^\mu{}_\alpha + \eta_{\alpha\nu}\varepsilon^\nu{}_\beta$

$$-\varepsilon^\mu{}_\alpha \eta_{\mu\beta} = \eta_{\alpha\nu}\varepsilon^\nu{}_\beta \qquad (-\varepsilon^T\eta = \eta\varepsilon)$$

xt-component:

$$+\varepsilon_{tx}\cancel{\eta_{tt}} = \cancel{\eta_{xx}}\varepsilon_{xt}$$

$$\varepsilon^\mu{}_\nu = \left(\begin{array}{c|ccc} 0 & \beta_x & \beta_y & \beta_z \\ \hline \beta_x & 0 & \theta_z & -\theta_y \\ \beta_y & -\theta_z & 0 & \theta_x \\ \beta_z & \theta_y & -\theta_x & 0 \end{array}\right)$$

3 rotations

3 boosts

Conclude: under translation ① $\frac{dx^\mu}{d\lambda}$ invariant, as $\frac{d^2x^\mu}{d\lambda^2}, \dots$

② & ③: invariants have all indices contracted!

$$\cancel{\frac{dx^\mu}{d\lambda}} \qquad \frac{dx^\mu}{d\lambda}\eta_{\mu\nu}\frac{dx^\nu}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda} \quad \checkmark$$

Now invoke "fake reparameterization": $\underbrace{\lambda \rightarrow f(\lambda)}$

$$S = \int d\lambda \; L\left(\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda}\right)$$

$\int df(\lambda) = f'(\lambda)\,d\lambda$

$$\frac{dx^\mu}{d\lambda} \rightarrow \frac{dx^\mu}{df(\lambda)} = \frac{dx^\mu}{d\lambda}\frac{d\lambda}{df(\lambda)} = \frac{1}{f'}\frac{dx^\mu}{d\lambda}$$

S mustn't change for any $f(\lambda)$. So

const.

$$S = \int d\lambda \, \cancel{f'(\lambda)} \frac{1}{\cancel{f'(\lambda)}} \sqrt{-\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda}}\,(-m)$$

Now, choose $\lambda = t$:

$$S = -m \int dt \sqrt{-\left[-\left(\frac{dt}{dt}\right)^2 + \dot{x}^2 + \dot{y}^2 + \dot{z}^2\right]}$$

$$= -mc^2 \int dt \sqrt{1 - \frac{\dot{x}^2 - \dot{y}^2 - \dot{z}^2}{c^2}}$$

Restore c: $(c \to \infty)$

Taylor expand in c: $L = -mc^2 + \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$