PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 7
Relativistic charged particles

September 11

Relativistic particle: translation: $x^\mu(\lambda) \to x^\mu(\lambda) + c^\mu$ ← const.

$x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$ if $c=1$

"fake time" parameterized the trajectory, as long as $t(\lambda)$ is increasing

Lorentz (boost & rotation): $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ (sum over ν)

where $\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta_{\mu\nu} = \eta_{\alpha\beta}$ where $\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Define: $\eta_{\mu\nu} x^\nu = x_\mu$.

Using translation/Lorentz & reparameterization $\lambda \to f(\lambda)$

$$S = -m \int d\lambda \sqrt{-\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda}}$$

const.

Goal: interactions of relativistic particles w/ E&M

Suppose background EM potential $A_\mu(x)$ ← define later

$$S[x^\mu] = \cdots + q\int \cancel{d\lambda}\, \frac{dx^\mu}{\cancel{d\lambda}} A_\mu(x)$$

↑ charge!

Review of relativistic E&M:

$$A_\mu = \begin{pmatrix} A_t \\ A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} -\varphi \\ A_x \\ A_y \\ A_z \end{pmatrix}$$

← electric/scalar potential (voltage)

$$\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t}$$
$$\vec{B} = \nabla \times \vec{A}$$

← $\vec{A}$: magnetic vector potential

Look at: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$ ← note: index moved "downstairs" in der.

$= -F_{\nu\mu}$ (antisymmetric)

$\binom{4}{2} = 6$ independent components...

e.g. $\partial_\mu J^\mu = \partial_t J^t + \partial_x J^x + \ldots$ is natural operation: divergence

$$F_{tt} = \partial_t A_t - \partial_t A_t = 0$$

$$F_{ti} = -F_{it} = \partial_t A_i - \partial_i A_t = \partial_t A_i + \partial_i \varphi = -E_i$$

↑ $i \in \{x,y,z\}$ only!

Levi-Civita

$$F_{ij} = \partial_i A_j - \partial_j A_i = \varepsilon_{ijk} B^k$$

$B_z = \partial_x A_y - \partial_y A_x$ (F_{xy})

$\varepsilon_{ijk} = 0$ if ≥ 2 indices same

$\varepsilon_{xyz} = +1$

$\varepsilon_{ijk} = -\varepsilon_{jik} = -\varepsilon_{ikj}$.

Hence:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

For now: no Maxwell's eqn... particle in given "background" fields.

EM is invariant under gauge transformations;

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x) \qquad \partial_\mu \Lambda = \frac{\partial \Lambda}{\partial x^\mu}$$

Physical observable is not A_μ, only $F_{\mu\nu}$ is:

$$F_{\mu\nu} \to F_{\mu\nu} + \cancel{\partial_\mu \partial_\nu \Lambda} - \cancel{\partial_\nu \partial_\mu \Lambda} \quad \text{is invariant!}$$

Return to point particle Lagrangian! ← "relativistic covariance"... NOT invariant (transform DOF x^μ & parameters A_μ)

Claim: $S = -m\int d\lambda \sqrt{-\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda}} + \underbrace{q\int d\lambda \frac{dx^\mu}{d\lambda} A_\mu}_{\text{reparameterization!}}$

$F^{\alpha\beta} = \eta^{\alpha\mu} F_{\mu\nu} \eta^{\nu\beta}$ and $\eta^{\mu\nu} = \eta_{\mu\nu}$

Why not add $\int d\lambda \frac{dx^\mu}{d\lambda} A_\mu \underbrace{F^{\alpha\beta} F_{\alpha\beta}}_{\text{gauge invariant!}}$?

This term $\frac{dx^\mu}{d\lambda} A_\mu$ transforms nicely under gauge:

$$\int d\lambda \frac{dx^\mu}{d\lambda}\left[A_\mu + \partial_\mu \Lambda(x)\right] = \int d\lambda \frac{dx^\mu}{d\lambda} A_\mu + \cancel{\int d\lambda \frac{d}{d\lambda}\Lambda(x)}$$

ignore total derivatives in L.

$$\int d\lambda \frac{dx^\mu}{d\lambda}\left[A_\mu + \partial_\mu \Lambda\right] F^{\alpha\beta} F_{\alpha\beta} = \cdots + \underbrace{\int d\lambda \frac{d\Lambda}{d\lambda} F^{\alpha\beta} F_{\alpha\beta}}_{\text{no longer total derivative...}}$$

Therefore: S (action) again completely fixed.

didn't need regime of validity... symmetry strong enough! (quite rare)

Equations of motion via Euler-Lagrange equations

also: pick $\lambda = t$... useful b/c now config space $\mathbb{R}^3_{(x,y,z)}$

$$S = -m\int dt\sqrt{1-\underbrace{\dot{x}^2-\dot{y}^2-\dot{z}^2}_{\dot{x}^j\dot{x}_j}} + q\int dt\left[A_t + \dot{x}^j A_j\right]$$

$A_t = -\varphi$

can depend on t, x_i

$$\frac{\delta S}{\delta x^i} = \frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0$$

change summed index in S to avoid confusion.

$\frac{\partial(\dot{x}^j A_j)}{\partial \dot{x}^i} = \delta^j_i A_j = A_i$

identity

$$\left[-q\frac{\partial\varphi}{\partial x^i} + q\dot{x}^j\frac{\partial A_j}{\partial x^i}\right] - \frac{d}{dt}\left[\frac{m\dot{x}_i}{\sqrt{1-\dot{x}_j\dot{x}^j}} + qA_i\right] = 0$$

$\frac{dA_i}{dt} = \frac{\partial A_i}{\partial t} + \frac{\partial A_i}{\partial x^j}\dot{x}^j$

$$m\frac{d}{dt}\underbrace{\left[\frac{\dot{x}_i}{\sqrt{1-\dot{x}_j\dot{x}^j}}\right]} = q\left[-\partial_i\varphi - \partial_t A_i\right] + q\dot{x}^j\underbrace{\left[\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}\right]}_{\partial_i A_j - \partial_j A_i}$$

$\frac{d}{dt}\left(\frac{mv_i}{\sqrt{1-v^2/c^2}}\right)$

$$= qE_i + q\dot{x}^j F_{ij}$$

$$= qE_i + q\dot{x}^j\varepsilon_{ijk}B^k$$

$$= q(\vec{E} + \vec{v}\times\vec{B})_i \qquad \text{where } \vec{v} = \frac{d\vec{x}}{dt}.$$

$\frac{d}{dt}(\text{mom.}) = \text{force}$

Reproduced Lorentz force equation! Relativistic Newton's 2nd Law!

Form of EM coupled to charged particles completely fixed!