PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 8
Configuration space
September 13

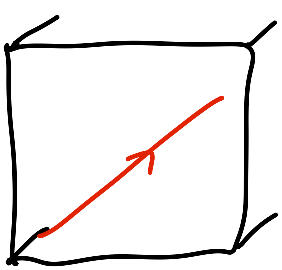
Config. space = set of all physically distinct... config...

↑ denote as X

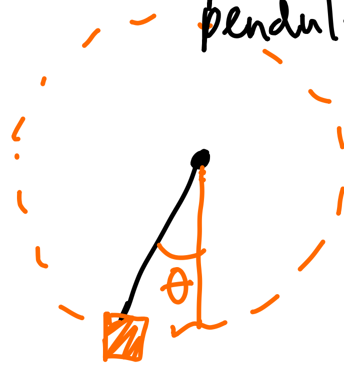
Example 1: relativistic particle

$$S = -m\int dt\,\sqrt{1-\dot{x}_i\dot{x}^i}$$

$$X = \mathbb{R}^3 \leftarrow (x,y,z) = x^i$$

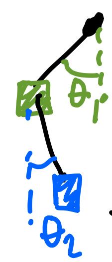


Example 2: pendulum



X = circle $= S^1$

Example 3: double pendulum



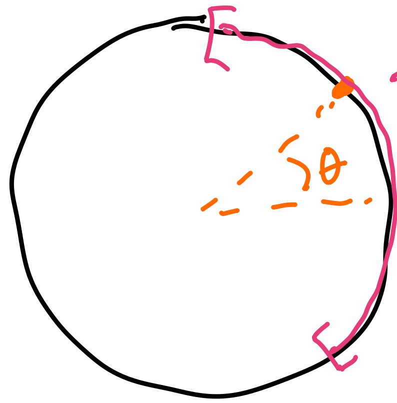
"join 2 smaller conf. spaces"

$$X = S^1 \times S^1$$
$$(\theta_1, \theta_2)$$

and many more...

Mathematically: config space = manifold (n-dimensional) = smooth space, no boundary, locally look like $\mathbb{R}^n$

The circle S^1:



→ look like (a piece of) $\mathbb{R}^1$.

...

globally $S^1 \neq \mathbb{R}$

b/c $\theta \sim \theta + 2\pi$

two config are indistinguishable

Formally, principle of least action is reasonable?

$$S[x_i(t)] = \int dt\, L(\underbrace{x_i, \dot{x}_i, \ldots}_{\text{how to correctly write down?}})$$

Proceed w/ as little as math as possible...

Strategy 1: guess clever (local) coordinates where X "looks like" $\mathbb{R}^n$

for S^1: write $L(\dot{\theta}, \theta)$

Demand $L(\dot{\theta}, \theta + 2\pi) = L(\dot{\theta}, \theta)$

} may only exist on part of X... OK if not looking at long Δt traj.

↓

POLA: $\dfrac{\delta S}{\delta \theta} = \dfrac{\partial L}{\partial \theta} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\theta}} = 0.$

Handle symmetries as before..

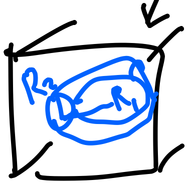
Strategy 2: embed X ← n-dim. in a higher-dimensional $\mathbb{R}^{n'}$ $(n' > n)$

↳ points in X solution to constraint eqs:

e.g. $\underbrace{S^1 \times S^1}_{\text{torus}}$

$$\underbrace{F_1(x_1, \ldots, x_{n'})}_{\mathbb{R}^{n'}} = \cdots = F_{n'-n}(x_1, \ldots, x_{n'}) = 0$$

each eq. removes 1 dimension



$\mathbb{R}^3$

Write Lagrangian:

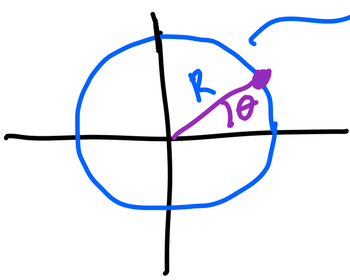
$$L(x_i, \dot{x}_i, \lambda_\alpha) = L_0(x_1, \ldots, \dot{x}_{n'}) + \sum_{\alpha=1}^{n'-n} \lambda_\alpha F_\alpha(x_1, \ldots, x_n)$$

$i = 1 \ldots n'$ | Lagrange multipliers | holonomic constraint

$$\frac{\delta S}{\delta \lambda_\alpha} = \frac{\partial L}{\partial \lambda_\alpha} = F_\alpha = 0.$$ (impose constraints)

Then $\frac{\delta S}{\delta x_i} = 0 \ldots$ (try to remove $\lambda_\alpha \ldots$)

For S^1:



$x^2 + y^2 - R^2 = 0.$

If solve? $\rightarrow$ Strategy 1

$x = R\cos\theta \qquad y = R\sin\theta$ new coord...

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \lambda(x^2 + y^2 - R^2)$$

$$\frac{\delta S}{\delta \lambda} = x^2 + y^2 - R^2 = 0$$

$$\frac{\delta S}{\delta x} = 2\lambda x - m\ddot{x} = 0 \quad \ldots \quad 0 = 2\lambda y - m\ddot{y}$$

Get rid of λ?

$$m(x\ddot{x} + y\ddot{y}) = 2\lambda(x^2 + y^2) = 2R^2\lambda$$

$$\rightarrow \lambda = \frac{m}{2R^2}\underbrace{(x\ddot{x} + y\ddot{y})}_{\text{alternate expression?}}$$

$$\frac{d}{dt}\left[x^2 + y^2 - R^2\right] = 0 = 2x\dot{x} + 2y\dot{y} = 0.$$

$\hookrightarrow \frac{d}{dt}$ again: $\dot{x}^2 + \dot{y}^2 = -(x\ddot{x} + y\ddot{y})$

$$m\ddot{x} = 2\lambda x = \frac{-m}{R^2}(\dot{x}^2 + \dot{y}^2)x$$

centripetal acceleration

$$m\ddot{x} = \underbrace{-m\frac{\dot{x}^2+\dot{y}^2}{R}\,\frac{x}{R}}$$ ← angular factor

right hand side: "normal force" required to enforce constraint.

Strategy 1 (again):

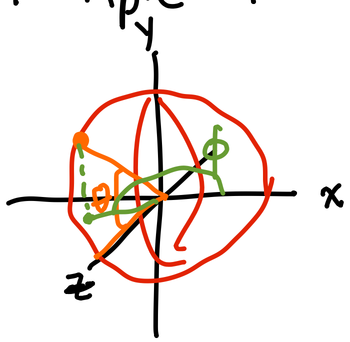
$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \lambda\underbrace{(x^2+y^2-R^2)}$$

$x = R\cos\theta \quad y = R\sin\theta$

→ Plug-in: $L = \frac{1}{2}mR^2\dot{\theta}^2 + \cancel{\lambda \cdot 0}$

Often, solving constraint is hard (Lec 9-10)

Example 4: S^2, the two-dimensional sphere



$$x^2 + y^2 + z^2 - R^2 = 0$$

Strategy 1:

$$x^2 + y^2 = R^2\sin^2\theta$$
$$z^2 = R^2\cos^2\theta$$

and

$$\left.\begin{aligned} x &= R\sin\theta\cdot\cos\phi \\ y &= R\sin\theta\cdot\sin\phi \end{aligned}\right\} \quad \phi \sim \phi + 2\pi$$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$\hookrightarrow \frac{1}{2}mR^2(\dot{\theta}^2 + \underbrace{\sin^2\theta\,\dot{\phi}^2})$$

$\sin^2\theta$ needed for consistency w/ $\theta = 0$ being one point