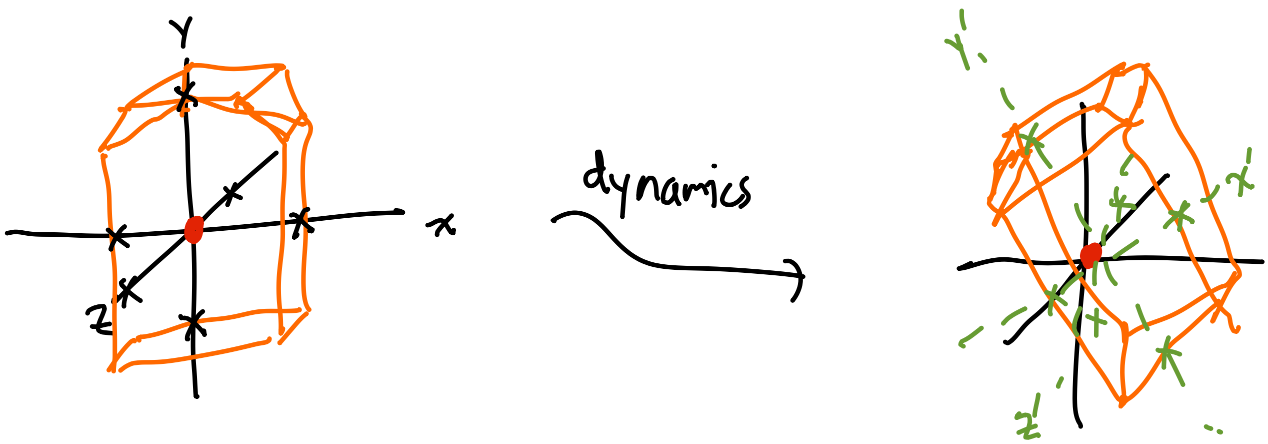
PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 9
Configuration space of a rigid body

September 16

X (config space) = manifold of physically distinguishable config/orientations.

Today: rigid body (1 pt fixed)
↳ HW 3 includes translational



space frame: (x, y, z)

body frame: (x', y', z')

$X = \{$rotations from body frame $\to$ space frame$\}$

How to describe (quantitatively) X?

↳ set of 3 orthonormal basis vector

in body frame: $\underbrace{\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}_{z'}^{x' \, y'}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}}_{\text{basis-dependent statement}} \xrightarrow{\text{abstract}} \vec{e}_I$

$I = 1,2,3$ to denote body frame.

$$\vec{e}_I \cdot \vec{e}_J = \delta_{IJ} = \begin{cases} 1 & \text{if } I = J \\ 0 & I \neq J \end{cases} \quad \text{and} \quad \vec{e}_I \cdot (\vec{e}_J \times \vec{e}_K) = \varepsilon_{IJK}$$

Now define space frame orthonormal coords:

$\vec{R}_i$. Again $\vec{R}_i \cdot \vec{R}_j = \delta_{ij}$ and $\vec{R}_i \cdot (\vec{R}_j \times \vec{R}_k) = \varepsilon_{ijk}$

$i = 1,2,3$ is space frame

To describe X, $\vec{R}_i$ in terms of $\vec{e}_J$, or:

$R_{iJ} = \vec{R}_i \cdot \vec{e}_J$ is a 3×3 matrix

(constrained by orthonormal...

$$\vec{R}_i \cdot \vec{R}_j = \delta_{ij} = \left(\sum_I R_{iI} \vec{e}_I \right) \cdot \left(\sum_J R_{jJ} \vec{e}_J \right) \to R_{iI} R_{jJ} \delta_{IJ}$$

$$\delta_{ij} = R_{iI} R_{jI}$$

$$(\mathbb{1} = R R^T)$$

3×3 matrix; define:

$$\text{So } X \to \{ R_{iI} : \underbrace{R_{iI} R_{jI} = \delta_{ij}}\ (R \text{ orthogonal}) \} = O(3)$$

$$\det(R^T)\det(R) = \det(R R^T) = \det(\mathbb{1}) = 1 = \det(R)^2. \text{ So } \det R = \pm 1.$$

$$R_{+1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \varepsilon_{IJK} = \vec{e}_I \cdot (\vec{e}_J \times \vec{e}_K) \text{ wrong}$$

$\det = 1$ $\qquad$ $\det = -1$ → parity-reversed universe

A continuous trajectory on $O(3)$ can't jump discontinuously from $\det(R) = +1$ to -1, so X should restrict to $+1$:

$$X = SO(3) = \{R \in O(3) \text{ w/ } \det R = +1\}.$$

$SO(3)$ itself is a Lie group (rotation)

↳ Lec 10: implies symmetry for physical Lagrangian ...

How many DOF? (manifold's dimension)

↳ locally look like $\mathbb{R}^n$ ← TBD, w/ fixed n.

↳ pick clever pt to find dimension.

identity $\in SO(3)$

↳ $R_{iI} = \underset{\uparrow}{\delta_{iI}} + \underset{\uparrow \text{ infinitesimal} \ldots}{\varepsilon_{iI}} + \mathcal{O}(\varepsilon^2)$

$$\delta_{iI} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Check: $R_{iI}R_{iJ} = \delta_{IJ}$ or $R_{iI}R_{jI} = \delta_{ij}$

$$\downarrow$$

$$(\delta_{iI} + \varepsilon_{iI})(\delta_{jI} + \varepsilon_{jI}) = \delta_{ij} + \mathcal{O}(\varepsilon^2)$$

$$\delta_{ij} + \underbrace{\varepsilon_{iI}\delta_{jI} + \delta_{iI}\varepsilon_{jI}}_{\varepsilon_{ij} + \varepsilon_{ji} \, = \, 0} + \ldots =$$

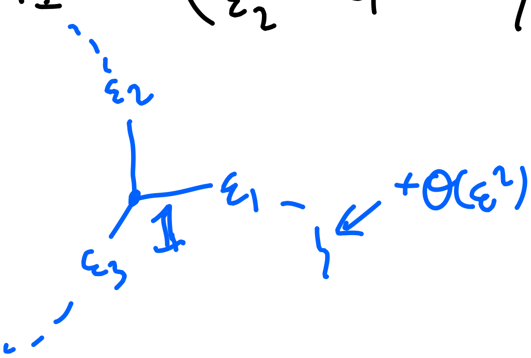
ε is antisymmetric ← $\varepsilon_{ij} + \varepsilon_{ji} = 0$

$$\varepsilon_{iI} \to \begin{pmatrix} 0 & \varepsilon_3 & -\varepsilon_2 \\ -\varepsilon_3 & 0 & \varepsilon_1 \\ \varepsilon_2 & -\varepsilon_1 & 0 \end{pmatrix}$$

→ 3 independent components $\varepsilon_{1,2,3}$

$$\downarrow$$

$$\dim(SO(3)) = 3.$$



Goal (lec 10): $L = L(R_{iI}, \dot{R}_{iI}, \dots)$

$$+ \Lambda_{IJ}(R_{iI}R_{iJ} - \delta_{IJ})$$

(Lagrange multiplier for orthogonality...)

Count: $\dim SO(3) = 3.$ $\quad \dim(\Lambda) = \dim(3\times 3 \text{ matrix}) = 9$

$= \dim(R)\dots$

How to remove only 6 DOF?

Look at: $R_{iI}R_{iJ} - \delta_{IJ} = 0 = R_{iJ}R_{iI} - \underset{= \delta_{IJ}}{\delta_{JI}}$

so: $\Lambda_{IJ}(\dots) = \boxed{\dfrac{\Lambda_{IJ} + \Lambda_{JI}}{2}}(R_{iI}R_{iJ} - \delta_{IJ})$

$$= \Lambda_{IJ} \frac{(R_{iI}R_{iJ} - \delta_{IJ}) + (R_{iJ}R_{iI} - \delta_{JI})}{2}$$

only symmetric part of Λ matters:

$\dim(\text{symmetric } 3\times 3) = \dim(3\times 3) - \dim(\text{antisym } 3\times 3)$

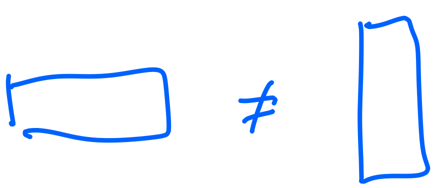
$= 9 - 3 = 6.$

Preview: universe's rotation symmetry...

... freedom to choose space frame. (left-SO(3) inv.)

$$R_{iI} \to Q_{ij}R_{jI}$$

but NO right-SO(3) → objects are asymmetric



(rotated by 90°)