Physics 7450, Fall 2019

2. Kinetic theory of transport

2.1) Boltzmann equation for a particle

Reference: Rammer, “Quantum transport theory” (2004)

Consider a Hamiltonian for a non interacting particle in one dimension (it is straightforward to generalize to higher dimensions)

e.g. $$H = \varepsilon(p) + V(x) \qquad p = -i\partial_x \quad (\hbar = 1)$$

If $V(x) = 0$, then

Exact solution: $$i\partial_t \Psi = \varepsilon(-i\partial_x)\Psi$$

$$\Rightarrow \Psi = \int dp\, e^{ipx}\, \Psi(p,t), \quad \Psi(p,t) = \Psi(p,0)\, e^{-i\varepsilon(p)t}$$

But this is not necessarily so enlightening as is. To try and get somewhere a little more physical we will introduce the Wigner transform

$$C(x,y,t) = \Psi(x,t)\,\overline{\Psi(y,t)}$$

$$f(x,p,t) = \int dz\; C\left(x+\frac{z}{2}, x-\frac{z}{2}, t\right) e^{-ipz} = \mathcal{W}(C)$$

$$C(x,y,t) = \int \frac{dp}{2\pi}\, e^{ip(x-y)}\, f\left(\frac{x+y}{2}, p\right)$$

To calculate the evolution of this function we will do so in a slightly convoluted fashion

let $\mathcal{W}(g_i) = G_i$; $g_1(x,y) = \int dz\; g_2(x,z)\, g_3(z,y)$

$$g_1(x,y) = \int dz\, \frac{dq\,dk}{(2\pi)^2}\, e^{ik(x-z)+iq(z-y)}\, G_2\left(\frac{x+z}{2}, k\right) G_3\left(\frac{y+z}{2}, q\right)$$

let $X = \frac{x+y}{2}$, $Y = x-y$;

$$G_1(X,p) = \int \frac{dz\,dY\,dq\,dk}{(2\pi)^2}\, e^{-ipY + ik\left(X+\frac{Y}{2}-z\right) + iq\left(z-X+\frac{Y}{2}\right)}\, G_2\left(\frac{X+\frac{Y}{2}+z}{2}, k\right) G_3\left(\frac{X-\frac{Y}{2}+z}{2}, q\right)$$

Now let

$$X_2 = \frac{Y}{2} + z - X \qquad p_2 = k - p$$

$$X_3 = z - \frac{Y}{2} - X \qquad p_3 = q - p$$

$$G_1(X, p) = \int \frac{dX_2\, dX_3\, dp_2\, dp_3}{(2\pi)^2}\, e^{i(X_2 p_3 - X_3 p_2)}\, G_2\!\left(X + \frac{X_2}{2}, p + p_2\right) G_3\!\left(X + \frac{X_3}{2}, p + p_3\right)$$

Now observe...

$$f(x+a) = f(x) + a\,\partial_x f(x) + \frac{a^2}{2}\partial_x^2 f(x) + \cdots$$

$$= e^{a\partial_x} f(x)$$

$$e^{i(X_2 p_3 - X_3 p_2)}\, G_2\!\left(X + \frac{X_2}{2}, p + p_2\right) G_3\!\left(X + \frac{X_3}{2}, p + p_3\right)$$

$$= e^{i(X_2 p_3 - X_3 p_2)}\, G_2(X, p)\, e^{\frac{1}{2}\overleftarrow{\partial}_X X_2 + \frac{1}{2}\overrightarrow{\partial}_X X_3 + p_2 \overleftarrow{\partial}_p + p_3 \overrightarrow{\partial}_p}\, G_3(X, p)$$

Now evaluate integrals over $p_{3,2}$ and $X_{3,2}$:

$$\int \frac{dX_2}{2\pi}\, e^{i X_2 p_3 + \frac{1}{2}\overleftarrow{\partial}_X X_2} = \delta\!\left(i p_3 + \frac{1}{2}\overleftarrow{\partial}_X\right)$$

$$= G_2(X, p)\, e^{\frac{i\hbar}{2}\left(\overleftarrow{\partial}_X \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_X\right)}\, G_3(X, p) = G_2 * G_3$$

where we went ahead and inserted $\hbar$.

So convolutions become replaced with this Moyal star product.

Now let's understand what the Hamiltonian does... using $H = \varepsilon(p) + V(x)\ldots$

$$H(x,y) = \left[\varepsilon(i\partial_x)\delta(x-y)\right] + V(x)\delta(x-y)$$

$$\hookrightarrow \mathcal{W}[H] = V(x) + \varepsilon(p)$$

We can finally evaluate the time evolution of our f

$$\partial_t f(x,p,t) = \frac{1}{\hbar}\mathcal{W}\left[-i(H\Psi)\bar{\Psi} + i\Psi(H\bar{\Psi})\right] = -\frac{i}{\hbar}\left(\mathcal{W}[H] \star f - f \star \mathcal{W}[H]\right)$$

At leading order this looks like

$$\partial_t f = \frac{\partial H}{\partial x}\frac{\partial f}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial f}{\partial x} + \mathcal{O}(\hbar^2)$$

group velocity: $v(p) = \frac{\partial H}{\partial p} = \frac{\partial \varepsilon}{\partial p}$, external force $F = -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x}$

$$\boxed{\partial_t f + v\,\partial_x f + F\,\partial_p f = 0}$$

This is the kinetic or Boltzmann equation for a non interacting system. It can be shown to also hold for a collection of non interacting degrees of freedom and also generalizes in an obvious way into higher dimensions.

Green's function: if $(\tilde{x}(t), \tilde{p}(t))$ solve Hamilton's equations w/ $(\tilde{x}(0), \tilde{p}(0))$ given: $f = \delta(x - \tilde{x}(t))\,\delta(p - \tilde{p}(t))$

higher order $\mathcal{O}(\hbar^2)$ corrections negligible when

$$\hbar \frac{\partial}{\partial x}\frac{\partial}{\partial p} \ll 1, \quad \text{or} \quad \boxed{\Delta x \Delta p \gg \hbar}$$

2.2) Particle in a random potential

Now we turn on a random potential V(x) and study the problem in higher dimensions

$$\partial_t f + \vec{v}\cdot\partial_{\vec{x}} f - \partial_{\vec{x}} V \cdot \partial_{\vec{p}} f = 0$$

$$\mathbb{E}\left[V(\vec{x})V(\vec{y})\right] = g(|\vec{x}-\vec{y}|).$$

average over disorder

Suppose g is very small, we can try to perturbatively integrate it out

At $\mathcal{O}(V^0)$: $\quad \partial_t f + \vec{v}\cdot\nabla_{\vec{x}} f = 0$

$$\Downarrow \text{Fourier}$$

$$(-i\omega + ik\cdot v) f = 0$$

At $\mathcal{O}(V)$:

$$-i\omega f^{(1)} + i\vec{k}\cdot\vec{v} f^{(1)} = \int \frac{d^d q}{(2\pi)^d}\, iV(\vec{q})\, \vec{q}\cdot \frac{\partial f^{(0)}(\vec{k}-\vec{q})}{\partial \vec{p}}$$

At $\mathcal{O}(V^2)$:

$$\int \frac{d^d q}{(2\pi)^d}\, iV(\vec{q})\left(\vec{q}\cdot\frac{\partial}{\partial\vec{p}}\right)\int \frac{d^d q'}{(2\pi)^d}\, \frac{iV(\vec{q}\,')}{i((\vec{k}'+\vec{q})\cdot\vec{v} - \omega)}\, \vec{q}\,'\cdot\frac{\partial f^{(0)}(\vec{k}')}{\partial\vec{p}} = (-i\omega + i\vec{k}\cdot\vec{v}) f^{(2)}$$

$$\vec{k} = \vec{q} + \vec{q}\,' + \vec{k}'$$

principal

Now use $\frac{i}{\omega} \to \mathcal{P}\frac{i}{\omega} + \pi\delta(\omega)$, and consider long time scales $(\omega\to 0)$

$$= -\pi\int\frac{d^d q\, d^d q'}{(2\pi)^{2d}}\, V(q)V(q')\left[\left(\vec{q}\cdot\frac{\partial}{\partial\vec{p}}\right)\left(\vec{q}\,'\cdot\frac{\partial}{\partial\vec{p}}\right)f^{(0)}\right]\delta(\vec{q}\,'\cdot\vec{v})$$

Disorder average: $\mathbb{E}\left[V(q)V(q')\right] = G(|\vec{q}|)\,\delta(\vec{q}+\vec{q}\,')\,(2\pi)^d$:

$$= \pi\int\frac{d^d q}{(2\pi)^d}\, G(q)\left[\left(\vec{q}\cdot\frac{\partial}{\partial\vec{p}}\right)^2 f\right]\delta(\vec{q}\cdot\vec{v})$$

In order to compare with Fermi's golden rule, consider ...

$$\delta(\vec{q}\cdot\vec{v}) \approx \delta\left(\varepsilon(\vec{q}+\vec{p}) - \varepsilon(\vec{p})\right) \approx \delta\left(\varepsilon(\vec{p}-\vec{q}) - \varepsilon(\vec{p})\right)$$

$$\left(\vec{q}\cdot\frac{\partial}{\partial\vec{p}}\right)^2 f \approx f(\vec{p}+\vec{q}) - 2f(\vec{p}) + f(\vec{p}-\vec{q}) .$$

Going back to real space and defining a transition rate

$$W(\vec{p}\,\vec{p}\,') = \frac{\pi}{\hbar}\delta\left(\varepsilon(\vec{p}) - \varepsilon(\vec{p}\,')\right)\left|V(\vec{p}-\vec{p}\,')\right|^2 \quad \swarrow \text{put } \hbar \text{ back in}$$

$$\partial_t f + \vec{v}\cdot\partial_{\vec{x}} f \approx \int\frac{d^d\vec{q}}{(2\pi\hbar)^d} W(\vec{p},\vec{p}+\vec{q})\Big(\underbrace{f(\vec{p}+\vec{q})}_{\text{ingoing}} - \underbrace{f(\vec{p})}_{\text{outgoing}}\Big)$$

The right hand side of this equation can be interpreted as the relative rate of scattering in vs scattering out particles. The scattering rate is given by Fermi's golden rule in the born approximation, which makes sense given we are looking at large length scales compared to quantum wavelengths.

Right hand side independent of x when $\partial_x f \ll \dfrac{f}{\underbrace{\Delta x_{\mathrm{dis}}}_{\text{disorder length scale.}}}$

2.3) Electrical conductivity from impurities

Reference: Ziman, "Electrons and phonons" (1960)

Now let us use the Boltzmann equation to start thinking about transport calculations. We start by thinking about electrical transport for electrons.

$$\partial_t f + v\cdot\nabla_x f + F\cdot\nabla_p f = -\int\frac{d^d p'}{(2\pi\hbar)^d} W(\vec{p},\vec{p}\,')\left(f(\vec{p}) - f(\vec{p}\,')\right)$$

In equilibrium: $(F=0)$ $\quad f_F(\vec{x},\vec{p}) = \dfrac{1}{1+e^{(\varepsilon(\vec{p})-\mu)/T}}$

μ = chemical potential, or Fermi energy

Check: $(\partial_t + v\cdot\nabla_x) f = 0$ by construction,

$$\int d^d p' \, \delta(\varepsilon(p) - \varepsilon(p')) \left|V(p-p')\right|^2 \left(f_F\left(\frac{\varepsilon(\vec{p})-\mu}{T}\right) - f_F\left(\frac{\varepsilon(\vec{p}\,')-\mu}{T}\right)\right) = 0!$$

Perturb by a small electric field:

$$\vec{F} = -e\vec{E}.$$

$$f(\vec{p}) = f_F(\vec{p}) - \left.\frac{\partial f_F}{\partial \varepsilon}\right|_{\vec{p}} \underbrace{\Phi(\vec{p},t)}_{\Phi\sim \vec{E}} + O(E^2)$$

After some algebra:

$$-\int \frac{d^d\vec{p}\,'}{(2\pi\hbar)^d} |V(\vec{p},\vec{p}\,')|^2 \pi\delta(\varepsilon(\vec{p})-\varepsilon(\vec{p}\,'))\left(\Phi(\vec{p}) - \Phi(\vec{p}\,')\right) = \partial_t \Phi(\vec{p}) + e\vec{E}\cdot\vec{v}$$

We may as well abstract to the following;

$$\partial_t \Phi(\vec{p}) + \int \frac{d^d\vec{p}\,'}{(2\pi\hbar)^d} \underbrace{\tilde{W}(\vec{p},\vec{p}\,')}_{\text{generic matrix}} \Phi(\vec{p}\,') = -e\vec{E}\cdot\vec{v}(\vec{p})$$

if inversion symmetry: $\tilde{W}(\vec{p},\vec{p}\,') = \tilde{W}(-\vec{p},-\vec{p}\,')$

if time reversal: $\tilde{W}(\vec{p},\vec{p}\,') = \tilde{W}(-\vec{p}\,',-\vec{p})$

inversion + time reversal $\Rightarrow$ $\tilde{W}(\vec{p},\vec{p}\,')$ is symmetric.

We also want: $\tilde{W}$ positive semi-definite if our phase is stable as this is essentially a thermodynamic constraint. (irreversibility)

Looking at DC transport... $\Phi(\vec{p},t) = \Phi(\vec{p})$.

$$\boxed{\Phi(\vec{p}) = -e\tilde{W}^{-1}(\vec{E}\cdot\vec{v}(\vec{p}))}$$

$(J_i = \sigma_{ij}E_j)$

$$\vec{J} = \int \frac{d^d\vec{p}}{(2\pi\hbar)^d}(-e)\vec{v}(\vec{p})f(\vec{p})$$

group velocity = quasiparticle velocity of carriers!

In equilibrium:

$$\vec{J}_{eq} = -e \int \frac{d^d \vec{p}}{(2\pi\hbar)^d} f_F \vec{v} = -e \int \frac{d^d \vec{p}}{(2\pi\hbar)^d} \frac{\partial \varepsilon}{\partial \vec{p}} f_F\left(\frac{\varepsilon - \mu}{T}\right) = 0 \quad \text{(total derivative)}$$

First order correction:

$$\vec{J} = -e \int \frac{d^d p}{(2\pi\hbar)^d} \left(-\frac{\partial f_F}{\partial \varepsilon}\right)_{\vec{p}} \vec{v}(\vec{p}) \Phi(\vec{p}). \qquad \text{If} \quad J_i = \sigma_{ij} E_j:$$

$$\sigma_{ij} = \int \frac{d^d p \, d^d p'}{(2\pi\hbar)^{2d}} e^2 v_i(p) \tilde{W}(p,p')^{-1} v_j(p') \left(-\frac{\partial f_F}{\partial \varepsilon}\right)_{\vec{p}}$$

Our first honest transport calculation! A few key points. Firstly the Fermi factor restricts the dynamics to close to the Fermi surface. Secondly, W was related to a scattering rate, so indeed conductivity goes as a scattering time. The weaker the scattering, the stronger the conductivity.

This result was general but let's now return to the special case of impurity scattering. If impurities are randomly placed point potentials, then

$$V_{imp}(\vec{x}) = \sum_{i=1}^{N} \underbrace{\tilde{V}(\vec{x} - \vec{x}_i)}_{\text{single impurity potential}}$$

Fourier transform: $$V(\vec{q}) = \int d^d\vec{x}\, e^{-i\vec{q}\cdot\vec{x}} V_{imp}(\vec{x}) = \tilde{V}(\vec{q}) \sum_{i=1}^{N} e^{-i\vec{q}\cdot\vec{x}_i}$$

If $\vec{x}_i$ are random: $\mathbb{E}[V(\vec{q})] = 0.$

$$\mathbb{E}[V(\vec{q})V(\vec{q}')] = \tilde{V}(\vec{q})\tilde{V}(\vec{q}')\,\mathbb{E}\left[\sum_{i,j} e^{-i(\vec{q}\cdot\vec{x}_i + \vec{q}'\cdot\vec{x}_j)}\right]$$

$$= \tilde{V}(\vec{q})\tilde{V}(\vec{q}')\,\underbrace{\mathbb{E}\left[\sum_i e^{-i(\vec{q}+\vec{q}')\cdot\vec{x}_i}\right]}_{= N \times \frac{1}{\mathrm{Vol}} \delta(\vec{q}+\vec{q}')}$$

well-behaved continuum limit...

Hence we find that if $\mathbb{E}\left[V(\vec{q})V(\vec{q}\,')\right] = G(\vec{q})\,\delta(\vec{q}+\vec{q}\,')$ . . .

$$G(\vec{q}) = n_{imp} \left|\tilde{V}(\vec{q})\right|^2$$

impurity density

Short range disorder: $\tilde{V}(\vec{q}) \approx V_0$ when $q\lambda_F \ll 1$.

This implies that

integrate to 0

$$\int \frac{d^d\vec{p}\,'}{(2\pi\hbar)^d} W(\vec{p},\vec{p}\,')\, v_j(\vec{p}\,') = n_{imp} V_0^2 \int \frac{d^d\vec{p}\,'}{(2\pi\hbar)^d} \frac{\pi}{\hbar}\delta(\varepsilon(\vec{p}) - \varepsilon(\vec{p}\,'))\left(v_j(\vec{p}) - v_j(\vec{p}\,')\right)$$

$$= \frac{\pi}{\hbar} n_{imp} V_0^2 \times \nu(\varepsilon(\vec{p}))\, v_j(p) = \frac{1}{\tau_{imp}(\varepsilon)} v_j(\vec{p})$$

density of states

More generally we will often assume the collision integral has this simple structure of just multiplying by a time scale (up to some zero modes). This is called the relaxation time approximation

$$\sigma_{ij} \approx \int \frac{d^d\vec{p}}{(2\pi\hbar)^d} e^2 \tau_{imp}^{(\varepsilon)} v_i(\vec{p}) v_j(\vec{p}) \left(-\frac{\partial f_F}{\partial \varepsilon}\right)_{\vec{p}}$$

At very low temperatures, $\left(-\frac{\partial f_F}{\partial \varepsilon}\right)_{\vec{p}} \approx \delta(\varepsilon(\vec{p}) - \mu)$,

$$\boxed{\sigma_{ij} = \int_{FS} \frac{d^d\vec{p}}{(2\pi\hbar)^d} e^2 \tau_{imp}\, v_i(\vec{p}) v_j(\vec{p})\, \delta(\varepsilon(\vec{p}) - \mu)}$$

This is a classic result — for short range interacting impurities, conductivity controlled by a single time scale and the geometry of the Fermi surface. For a typical metal, the Fermi temperature is absurdly high, so neglecting thermal smearing of the Fermi surface is certainly justified. But in low density semiconductors this need not be the case.

usually $\underline{T_F \sim 10^4 \text{ K}}$

Assume that $\varepsilon(\vec{p}) = \frac{\vec{p}^2}{2m}$: $\mu = p_F^2/2m$ ...

$$\sigma_{ij} = \int \frac{d^d p}{(2\pi\hbar)^d} \left(\frac{\delta(p-p_F)}{v_F} \right) \frac{p_i p_j}{m^2} e^2 \tau_{imp}$$

$$\propto \frac{p_F^d}{\hbar^d m} e^2 \tau_{imp} \propto \frac{n e^2 \tau_{imp}}{m} !$$

So in this simplifying limit we essentially recover the Drude formula

2.4) Kinetic transport formalism

Next our goal is to essentially just recast some earlier derivations in a more formal framework, which will greatly aid as we start to turn on more complicated scattering mechanisms ...

Functions $\Phi(\vec{p})$ form a <u>vector space</u>:

$$|\Phi\rangle = \int d^d p \, \Phi(\vec{p}) |\vec{p}\rangle$$

Give the space the inner product $\langle \vec{p} | \vec{p}' \rangle = \left(-\frac{\partial f_F}{\partial \varepsilon} \right)\bigg|_{\vec{p}} \frac{\delta(\vec{p}-\vec{p}')}{(2\pi\hbar)^d}$.

Define collision matrix W such that $W|\Phi\rangle = \int d^d \vec{p}' \, \tilde{w}(\vec{p}, \vec{p}') |\vec{p}'\rangle \Phi(\vec{p}')$.

Now observe: If we define $|J_i\rangle = -e \int d^d \vec{p} \, \vec{v}(\vec{p}) |\vec{p}\rangle$, then the current is $J_i = \langle J_i | \Phi \rangle$. Moreover ...

$$\partial_t |\Phi\rangle = -W|\Phi\rangle + E_i |J_i\rangle \ldots$$

$$\boxed{\sigma_{ij} = \langle J_i | W^{-1} | J_j \rangle}$$

The heat current: $Q = J_E - \mu \frac{J}{-e}$, encoded by

$$|Q_i\rangle = \int d^d\vec{p}\, \left(\varepsilon(\vec{p}) - \mu\right) v_i(\vec{p})\, |\vec{p}\rangle .$$

What external force implies a temperature gradient?

$$\left(\vec{v}\cdot\nabla_x + \vec{F}_T\cdot\nabla_p\right) n_F\left(\frac{\varepsilon(\vec{p})-\mu}{T(x)}\right) = 0 ?$$

$$n_F' \times \left\{ -\frac{\vec{v}\,(\varepsilon(\vec{p})-\mu)}{T^2}\cdot\nabla T + \vec{F}_T\cdot\frac{\vec{v}}{T} \right\} = 0 \Rightarrow \vec{F}_T = \frac{\nabla T}{\cancel{T}}\left(\varepsilon(\vec{p})-\mu\right)$$

$$\Longrightarrow \partial_t|\Phi\rangle + W|\Phi\rangle = E_i|J_i\rangle - \frac{\nabla_i T}{T}|Q_i\rangle$$

As expected, the temperature gradient drives a heat current. We immediately generalize the previous arguments and obtain

$$\begin{pmatrix} \sigma_{ij} & T\alpha_{ij} \\ T\alpha_{ji} & T\bar{\kappa}_{ij} \end{pmatrix} = \begin{pmatrix} \langle J_i| \\ \langle Q_i| \end{pmatrix} W^{-1} \begin{pmatrix} |J_j\rangle & |Q_j\rangle \end{pmatrix} \; !$$

There are a few more universal facts we will learn independent of W. First, let's understand positivity of W...

Entropy production:

$$s[f(\vec{p})] = -\int \frac{d^d p}{(2\pi\hbar)^d} \left\{ f(\vec{p})\log f(\vec{p}) + (1-f(\vec{p}))\log(1-f(\vec{p})) \right\}$$

$$\frac{ds}{dt} = -\int \frac{d^d p}{(2\pi\hbar)^d} \left\{ \frac{\partial f}{\partial t}\log f - \frac{\partial f}{\partial t}\log(1-f) \right\}$$

$$= -\int \frac{d^d p}{(2\pi\hbar)^d} \left(-\frac{\partial f_F}{\partial \varepsilon}\right)_p (W\Phi)(p) \left\{ \underbrace{\log\frac{f_F}{1-f_F}}_{\frac{\varepsilon(p)-\mu}{T}} + \frac{1}{\cancel{f_F(1-f_F)}}\cancel{\left(-\frac{\partial f_F}{\partial \varepsilon}\right)}^{1/T} \Phi + \ldots \right\}$$

Since energy and charge are conserved...

$$-\int\frac{d^dp}{(2\pi\hbar)^d}(\varepsilon-\mu)\left(-\frac{\partial f_F}{\partial\varepsilon}\right)_p\Phi = \int\frac{d^dp}{(2\pi\hbar)}\frac{\partial}{\partial t}(\varepsilon f-\mu f)$$

$$=\frac{d}{dt}\Big[\underbrace{\int\frac{d^dp}{(2\pi\hbar)^d}\varepsilon f}_{\text{total energy}} - \mu\underbrace{\int\frac{d^dp}{(2\pi\hbar)^d}f}_{\text{total "charge"/number density}}\Big]=0.$$

So $T\dfrac{ds}{dt}=\int\dfrac{d^dp}{(2\pi\hbar)^d}\Phi(p)\tilde{W}(p,p')\Phi(p')\left(-\dfrac{\partial f_F}{\partial\varepsilon}\right)_p=\langle\Phi|W|\Phi\rangle$

second law of thermo $\Longrightarrow$ positivity of W.

Next, we discuss a very useful variational principle...

Theorem: let $\rho=\dfrac{1}{\sigma_{xx}}$. Then $\rho\le\dfrac{\langle\Phi|W|\Phi\rangle}{\langle J_x|\Phi\rangle^2}$ for any $|\Phi\rangle$.

Proof: let $|\bar\Phi\rangle$ be true solution of $\overset{\text{set to 1}}{E_x}|J_x\rangle=W|\bar\Phi\rangle$.
let $|\Phi\rangle=|\bar\Phi\rangle+|\tilde\Phi\rangle$.

Let $R[\Phi]=\dfrac{\langle\Phi|W|\Phi\rangle}{\langle J_x|\Phi\rangle^2}$. Note $R[\Phi]=R[\underset{\text{constant }\lambda\neq0}{\lambda}\times\Phi]$.

Therefore we can always write $|\Phi\rangle=|\bar\Phi\rangle+|\tilde\Phi\rangle$ where we take $\langle\tilde\Phi|J_x\rangle=0$ and $|\bar\Phi\rangle$ solves $W|\bar\Phi\rangle=\alpha|J_x\rangle$ for some α. Now:

$$R[\Phi]=\frac{\langle\bar\Phi|W|\bar\Phi\rangle+2\langle\bar\Phi|W|\tilde\Phi\rangle+\langle\tilde\Phi|W|\tilde\Phi\rangle}{\langle J_x|\bar\Phi\rangle^2}=\frac{\alpha\langle J_x|\bar\Phi\rangle+2\overset{0}{\cancel{\langle\tilde\Phi|J_x\rangle}}+\langle\tilde\Phi|W|\tilde\Phi\rangle}{\langle J_x|\bar\Phi\rangle^2}$$

Now scale $|\Phi)$ so $\alpha=1$ ...

$$R = \frac{1}{\langle J_x|W^{-1}|J_x)} + \frac{\langle\Phi|W|\Phi\rangle}{\langle J_x|W^{-1}|J_x)^2}$$

$$\geq \frac{1}{\sigma_{xx}} = \rho$$

$\blacksquare$

Corollary (Mattheisen's Rule Revisited): Let

$$W = W_1 + W_2.$$

Let $\rho_{1,2}$ be resistivity from scattering 1,2. Then $\rho \geq \rho_1 + \rho_2$.

<u>Proof</u>: Let $|\Phi)$ be minimizer of $R[\Phi]$. Then

$$\rho = \frac{\langle\Phi|W|\Phi\rangle}{(J_x|\Phi)^2} = \underbrace{\frac{\langle\Phi|W_1|\Phi)}{\langle J_x|\Phi)^2}}_{\geq \rho_1} + \underbrace{\frac{\langle\Phi|W_2|\Phi\rangle}{\langle J_x|\Phi\rangle^2}}_{\geq \rho_2}$$

$\blacksquare$

Corollary: adding a new scattering mechanism increases resistivity.
Proof: Above. Since $\rho_2 \geq 0$, $\rho \geq \rho_1$.

This is a great example of a theorem which has a very important physical loophole. We will see later in this course how adding a scattering mechanism can decrease the resistivity, simply because this kinetic formalism need not be applicable.

2.5) Thermal transport of electrons at low temperatures

We now return to a practical problem — given impurity scattering, what is the thermoelectric conductivity matrix?

$$\langle Q_x | W^{-1} | Q_x \rangle = \int \frac{d^dp}{(2\pi\hbar)^d} \left(-\frac{\partial f_F}{\partial \varepsilon}\right) v_i(p) v_j(p) (\varepsilon(p)-\mu)^2 \underbrace{\tau_{imp}(\varepsilon)}_{\text{same calculation as before, but keep } \varepsilon\text{-dependence...}}$$

$$\text{Identity: } \left(-\frac{\partial f_F}{\partial \varepsilon}\right) \approx \delta(\varepsilon-\mu) + \frac{\pi^2 T^2}{3} \delta''(\varepsilon-\mu) + O(T^4)$$

$$\langle Q_x | W^{-1} | Q_x \rangle = \int \frac{d^dp}{(2\pi\hbar)^d} v_i v_j \tau_{imp} \left[\delta(\varepsilon-\mu) + \frac{\pi^2 T^2}{3} \delta''(\varepsilon-\mu) + \cdots\right] (\varepsilon-\mu)^2$$

$$T \bar{\kappa}_{xx} \approx \frac{\pi^2 T^2}{3} \frac{\sigma_{xx}}{e^2}.$$

$$\langle J_x | W^{-1} | Q_x \rangle = \int \frac{d^dp}{(2\pi\hbar)^d} v_i v_j \tau_{imp} \left[\delta(\varepsilon-\mu) + \frac{\pi^2 T^2}{3} \delta''(\varepsilon-\mu) + \cdots\right] (\varepsilon-\mu)$$

$$\frac{\pi^2 T^2}{3} \int \frac{d^dp}{(2\pi\hbar)^d} \delta'(\varepsilon-\mu) v_i v_j \tau_{imp} = -\frac{\pi^2 T^2}{3} \frac{\partial}{\partial \mu} \frac{\sigma}{e}$$

$$\boxed{\alpha_{xx} = -\frac{\pi^2 T}{3e} \frac{\partial \sigma}{\partial \mu}}$$

Mott's Law

$$\kappa_{xx} = \bar{\kappa}_{xx} - \frac{T \alpha_{xx}^2}{\sigma_{xx}} = \bar{\kappa}_{xx} + O(T^3), \text{ so}$$

$$\boxed{\kappa_{xx} \approx \frac{\pi^2 T}{3e^2} \sigma_{xx}}$$

Wiedemann-Franz Law

Defining $\mathcal{L}_0 = \frac{\pi^2}{3e^2}$, experimentalists usually report $\mathcal{L} = \frac{\kappa}{T\sigma}$, or $\frac{\mathcal{L}}{\mathcal{L}_0}$

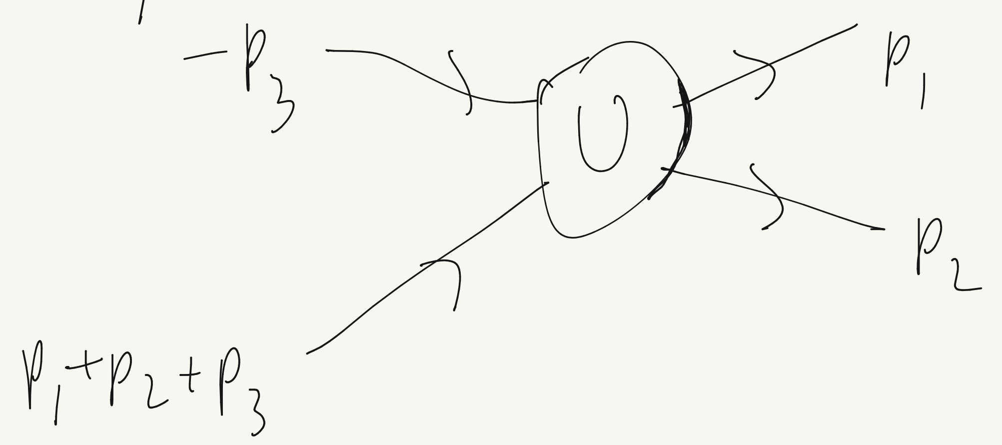
2.6) Electron-electron scattering

Now we turn to our first interaction effect — what happens if there are electron-electron interactions? The answer is that we need to add a new term to W, the collision integral...

A heuristic cartoon. Consider a sea of thermal electrons, and a single excitation moving around in this background...

$$H = \underbrace{\sum_{p} \varepsilon(\vec{p})\, c^{\dagger}(\vec{p})\, c(\vec{p})}_{\text{streaming terms}} + \sum_{p_1, p_2, p_3} U(p_1, p_2, p_3)\, c^{\dagger}(p_1)\, c^{\dagger}(p_2)\, c(p_3)\, c(-p_1-p_2-p_3)$$

Try to use Fermi's Golden Rule:



occurs at rate

$$\pi\, \delta\big(\varepsilon(p_1)+\varepsilon(p_2)-\varepsilon(p_3)-\varepsilon(p_1+p_2+p_3)\big)\, |U|^2$$

$$\times\ \underbrace{f(-p_3)\, f(p_1+p_2+p_3)}_{\text{prob. that incoming states occupied}} \underbrace{(1-f(p_1))(1-f(p_2))}_{\text{prob. that outgoing states unoccupied.}}$$

Boltzmann equation reads

$$\partial_t f + v\cdot\nabla_x f + F\cdot\nabla_p f = \int \frac{d^d p_1\, d^d p_2\, d^d p_3}{(2\pi\hbar)^{3d}}\, \pi\, \delta\Big(\sum \varepsilon\Big)\, \delta\Big(\sum \vec{p}\Big)\, |U|^2 \Big\{ \overbrace{(1-f(p))(1-f(p_1))\, f(p_2)\, f(p_3)}^{\text{incoming}}$$

$$- \underbrace{f(p)\, f(p_1)(1-f(p_2))(1-f(p_3))}_{\text{outgoing}} \Big\}$$

For transport, we only need to evaluate the linearized collision integral

Taylor expanding

$$f(p) \to f_F(p) - \frac{\partial f_F}{\partial \varepsilon} \Phi(p) + \cdots$$

At $\mathcal{O}(\Phi^0)$:

$$f_F(p) f_F(p_3)(1-f_F(p_2))(1-f_F(p_1)) = \frac{e^{\beta\varepsilon(p_1)+\beta\varepsilon(p_2)}}{(1+e^{\beta\varepsilon(p)})(1+e^{\beta\varepsilon(p_1)})(1+e^{\beta\varepsilon(p_2)})(1+e^{\beta\varepsilon(p_3)})}$$

By energy conservation...

$$= f_F(p_2) f_F(p_1)(1-f_F(p_3))(1-f_F(p)) \; !$$

So $\mathcal{O}(\Phi^0)$ terms vanish. At $\mathcal{O}(\Phi^1)$:

$$-\frac{\partial f_F}{\partial \varepsilon} = \frac{1}{T} f_F (1-f_F).$$

$$-f f_3 (1-f_1)(1-f_2) + f_1 f_2 (1-f)(1-f_3)$$

$$\to \frac{-1}{T}\left[f(1-f) f_3 (1-f_1)(1-f_2) + f_1 f_2 (1-f_3)(1-f) f\right] \Phi(\vec{p}) + 3 \text{ similar terms}$$

$$= \frac{-1}{T} f f_3 (1-f_1)(1-f_2)\left[\Phi(\vec{p}) + \Phi(\vec{p}_3) - \Phi(\vec{p}_1) - \Phi(\vec{p}_2)\right]$$

We conclude

$$\langle \Phi | W | \Phi \rangle = \pi\beta \int \frac{d^d p_1 \, d^d p_2 \, d^d p_3 \, d^d p}{(2\pi\hbar)^{4d}} \delta\left(\sum \varepsilon\right) \delta\left(\sum p\right) \Phi(p) \left[\Phi(p) + \Phi(p_3) - \Phi(p_1) - \Phi(p_2)\right]$$

$$\times |U|^2 f_F(p) f_F(p_3)(1-f_F(p_1))(1-f_F(p_2)).$$

In general, evaluating this object is quite nasty, but the temperature dependence is universal in a Fermi liquid

Consider trick:

$$\int \frac{d^d p}{(2\pi\hbar)^d} \longrightarrow \int \frac{d\varepsilon}{v} \int_{FS(\varepsilon)} d\Omega$$, and do ε-integrals. Assume $\Phi, |U|^2$ and v weakly depend on ε...

$$\int d\varepsilon\, d\varepsilon_3\, d\varepsilon_1\, d\varepsilon_2\; \delta(\varepsilon+\varepsilon_3-\varepsilon_1-\varepsilon_2)\cdots = \int d\omega\, d\varepsilon\, d\varepsilon_3\, d\varepsilon_1 d\varepsilon_2\; \delta(\omega+\varepsilon_1-\varepsilon)\, \delta(\omega+\varepsilon_3-\varepsilon_2)\cdots$$

If $|U|^2 \approx$ independent of ε near the Fermi surface...

$$\int d\varepsilon\, d\varepsilon_1 \frac{e^{\beta\varepsilon_1}}{(1+e^{\beta\varepsilon})(1+e^{\beta\varepsilon_1})}\delta(\omega+\varepsilon_1-\varepsilon)$$

$$= \int_{-\infty}^{\infty} d\varepsilon \frac{e^{\beta(\varepsilon-\omega)}}{(1+e^{\beta\varepsilon})(1+e^{\beta(\varepsilon-\omega)})} = \int_0^\infty \frac{dz}{\beta} \frac{e^{-\beta\omega}}{(1+z)(1+e^{-\beta\omega}z)}.$$ let $a = e^{-\beta\omega}$:

$$= \int_0^\infty \frac{dz}{\beta}\left[\frac{1}{1-a}\frac{1}{1+x} - \frac{a}{1-a}\frac{1}{1+ax}\right] = \frac{1}{\beta(1-a)}\log\frac{1}{a} = \frac{\omega}{1-e^{-\beta\omega}}$$

Doing a similar integral for $\varepsilon_2, \varepsilon_3$,

$$\beta\int_{-\infty}^{\infty} d\omega \frac{\omega^2}{(1-e^{-\beta\omega})^2}\, \Phi^2 |U|^2 \approx \Phi^2|U|^2 \int_{-\infty}^{\infty} d\omega \frac{\beta\omega^2}{(1-e^{-\beta\omega})^2}$$

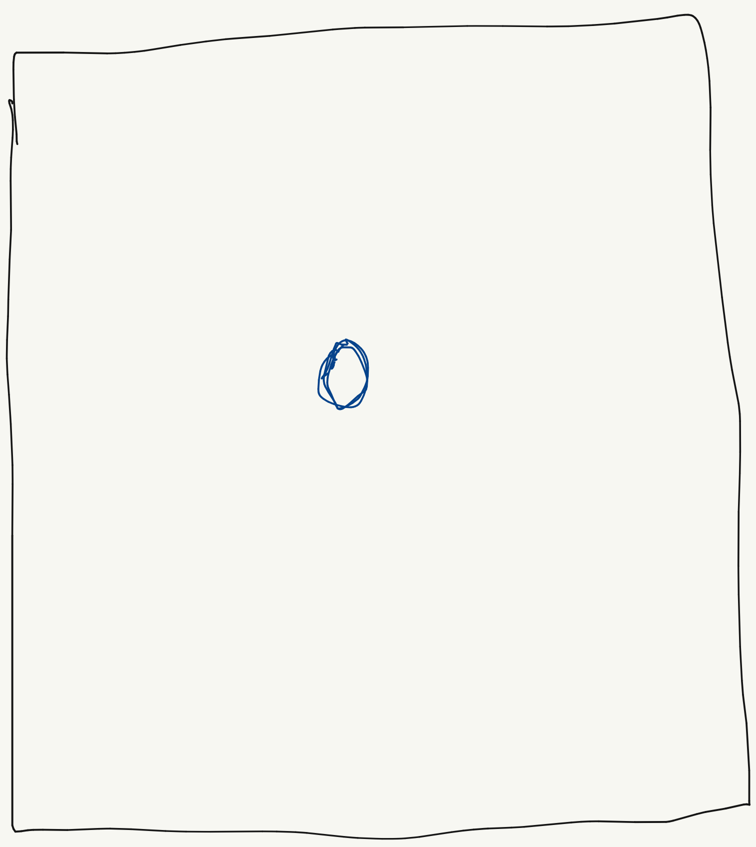
$$\propto \Phi^2 |U|^2 \times \underline{T^2}$$

We conclude that the electron-electron scattering rate is T^2. Note that in 2d there is a much richer story about the structure of the collision integral that we won’t discuss here....

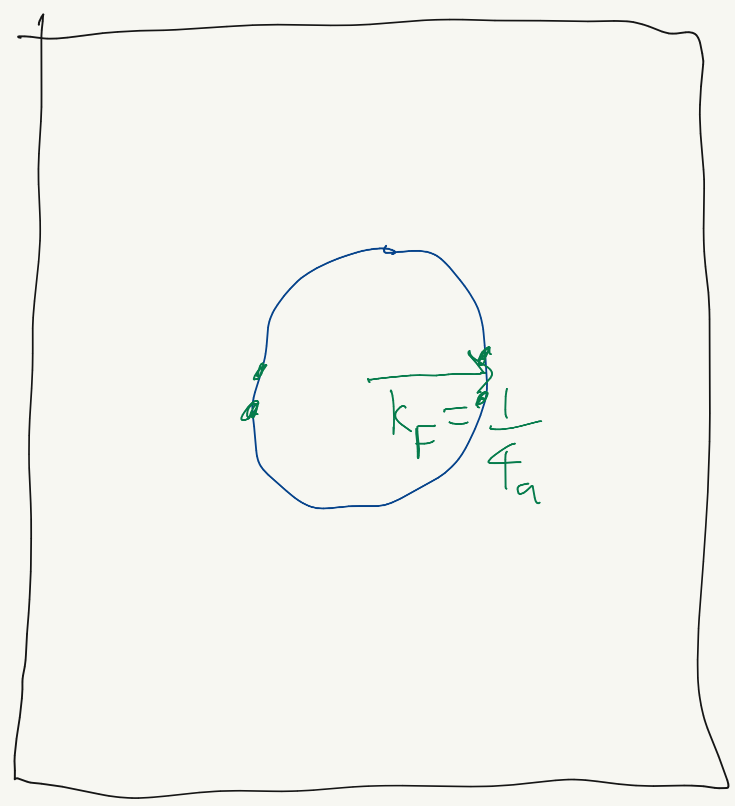
Now, it is tempting to generalize our previous argument about disorder scattering, simply replacing the impurity scattering time with this T^2. But there is a very important caveat....let's compute the decay time for momentum.

$$|P_i\rangle = \int d^d p \; p_i |\vec{p}\rangle .$$

$$\langle \Phi | W | P_j \rangle = \int \frac{d^d p \cdots d^d p_3}{(2\pi\hbar)^{4d}} |U|^2 \, \Phi(\vec{p}) \, \delta\left(\sum \varepsilon\right) \delta\left(\sum p\right) \underbrace{\left(p_j + p_{3j} - p_{1j} - p_{2j}\right)}_{= 0 ??} \times \cdots$$



basically all 2-body scattering momentum conserving...



reciprocal lattice vector

$$2 \times \frac{1}{4a} = 2 \times \left(-\frac{1}{4a}\right) + \frac{1}{a}$$

umklapp allowed.

Since $\delta\left(\sum p\right)$ is actually $\sum_{\vec{b}} \delta\left(\sum \vec{p} + \vec{b}\right)$ any reciprocal lattice vector...

momentum does decay w/ e-e umklapp!

Most metals have a large Fermi surface because each atom is contributing about 1 electron to the conduction band. So umklapp is usually there. But in semiconductors (GaAs) or graphene it is possible to have such a small Fermi surface that umklapp can be neglected

If FS very small...

$$\langle \Phi | W | P_i \rangle \sim \int (\Delta p_j) \times e^{-\beta(1/4a)} \sim e^{-T_0/T} \Rightarrow \text{umklapp negligible}$$

2.7) Thermoelectric transport with a small Fermi surface

In this section we explore what happens if there is a large discrepancy between umklapp and momentum conserving scattering rates. For simplicity, we consider a model with a circular Fermi surface.
Reference: 1804.00665

Note that in any Fermi liquid with a symmetry group G, the collision integral cannot mix sectors of different symmetry. So with rotational invariance we can restrict our study to

$$|\tilde{n}\rangle = \int \frac{dp \times p^{d-1} d\Omega}{(2\pi\hbar)^d} \cos\theta \, (|\vec{p}| - p_F)^n \, |\vec{p}\rangle$$

E.g. in $d=2$

$$\begin{Bmatrix} |J_x\rangle \\ |Q_x\rangle \\ |P_x\rangle \end{Bmatrix} = \int \frac{dp\, d\theta}{(2\pi\hbar)^2} \, p \times \cos\theta \begin{Bmatrix} -e\, v(p) \\ v(p)(\varepsilon(p)-\mu) \\ p \end{Bmatrix} |\vec{p}\rangle$$

all discrepancy is in radial dependence.

In this polynomial basis:

$$|J_x\rangle = -e\left(v_F|\tilde{0}\rangle + v_F'|\tilde{1}\rangle + \cdots\right) \quad \text{where } v_F' = v'(p_F).$$

$$|P_x\rangle = p_F|\tilde{0}\rangle + |\tilde{1}\rangle \qquad \text{(exact)}$$

$$|Q_x\rangle = v_F^2|\tilde{1}\rangle + \frac{3}{2}v_F v_F'|\tilde{2}\rangle + \cdots$$

This basis is not orthonormal... define

$$|0\rangle = \frac{|\tilde{0}\rangle}{\sqrt{\langle\tilde{0}|\tilde{0}\rangle}}; \qquad \langle\tilde{0}|\tilde{0}\rangle = \int \frac{dp\, d\theta}{(2\pi\hbar)^2} \cos^2\theta \left(-\frac{\partial f}{\partial \varepsilon}\right)$$

and since $dp = \frac{d\varepsilon}{v}$ ...

$$\approx \int \frac{d\varepsilon}{4\pi\hbar^2} \frac{p_F}{v_F}\left(-\frac{\partial f}{\partial \varepsilon}\right) = \frac{p_F}{4\pi\hbar^2 v_F}$$

We can interpret this as the density of states, like before:

$$n(\mu) = \frac{p_F^2}{4\pi\hbar^2} \; ; \qquad \nu = \frac{\partial n}{\partial \mu} = \frac{p_F}{2\pi\hbar^2}\frac{\partial p_F}{\partial \mu} = \frac{p_F}{2\pi\hbar^2 v_F}, \text{ so}$$

$$\langle \tilde{0} | \tilde{0} \rangle = \frac{1}{2}\nu + \mathcal{O}\left(\frac{T^2}{T_F^2}\right)$$

$$\langle \tilde{1} | \tilde{1} \rangle \simeq \frac{1}{2}\nu \times \frac{\pi^2}{3}\frac{T^2}{v_F^2} = \langle \tilde{0} | \tilde{2} \rangle \text{ (using identities previously derived)}$$

$$\langle \tilde{0} | \tilde{1} \rangle = \mathcal{O}(T^2) \ldots \text{ we won't need explicit expression}$$

$$|J_x\rangle \simeq -e\sqrt{\frac{\nu}{2}}\left[v_F|0\rangle + v_F' \frac{\pi T}{\sqrt{3}v_F}|1\rangle\right]$$

$$|Q_x\rangle \simeq \sqrt{\frac{\nu}{2}}\left[\frac{\pi T}{\sqrt{3}}v_F|1\rangle + \frac{\pi^2 T^2}{3 p_F}|0\rangle\right]$$

$$|P_x\rangle \simeq \sqrt{\frac{\nu}{2}}\left[p_F|0\rangle + \frac{\pi T}{\sqrt{3}v_F}|1\rangle\right]$$

using

$$\langle 0 | Q_x \rangle = \int \frac{dp\, p}{4\pi\hbar^2}(\varepsilon-\mu)v\frac{\pi^2 T^2}{3}\delta''(\varepsilon-\mu)$$

$$= \int \frac{d\varepsilon}{4\pi\hbar^2}\, p(\varepsilon-\mu)\frac{\pi^2 T^2}{3}\delta''(\varepsilon-\mu)$$

$$= \frac{\pi^2 T^2}{3}\frac{1}{4\pi\hbar^2}\left.\frac{\partial p}{\partial \varepsilon}\right|_{\varepsilon=\mu}$$

$$= \frac{\pi^2 T^2}{3} \times \frac{\nu}{2p_F}.$$

Now we need to estimate the form of the collision integral. First we start with electron-impurity scattering

$$\langle \tilde{n} | W_{imp} | \tilde{m} \rangle = \int \frac{dp}{4\pi\hbar^2}\, p(p-p_F)^{n+m}\,\hat{W}_{imp}(p)\left(\delta(\varepsilon-\mu) + \frac{\pi^2 T^2}{3}\delta''(\varepsilon-\mu) + \cdots\right)$$

$$\langle 0 | W_{imp} | 0 \rangle \equiv \frac{\nu}{2}\Gamma$$

$$\langle 0 | W_{imp} | 1 \rangle = \sqrt{\frac{\pi^2 T^2}{3 v_F^2}}\left(\frac{\partial \Gamma}{\partial p_F}\right)$$

$$\langle 1 | W_{imp} | 1 \rangle = \frac{\nu}{2}\Gamma$$

$$= \frac{\pi T}{\sqrt{3}}\frac{\partial \Gamma}{\partial \mu}$$

Now we turn to electron electron scattering, which we assume is momentum conserving for simplicity...

$$W_{ee} \equiv \gamma\left(1 - |P_x\rangle\langle P_x|\right) \approx \gamma\left(|1\rangle - \frac{\pi T}{\sqrt{3}p_F v_F}|0\rangle\right)\left(\langle 1| - \frac{\pi T}{\sqrt{3}p_F v_F}\langle 0|\right)$$

$$W^{-1} = (W_{ee} + W_{imp})^{-1}$$

$$\approx \frac{1}{\Gamma}|0\rangle\langle 0| - \frac{\pi T}{\sqrt{3}v_F}\frac{\Gamma' - \gamma/p_F}{\Gamma(\Gamma+\gamma)}\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right) + \frac{\Gamma + \frac{\pi^2 T^2}{3p_F^2 v_F^2}\gamma}{\Gamma(\Gamma+\gamma)}|1\rangle\langle 1|$$

Evaluate conductivities to lowest order in T AND in γ:

$$\sigma_{xx} = \langle J_x|W^{-1}|J_x\rangle \approx \frac{\nu}{2}\frac{e^2 v_F^2}{\Gamma}$$

$$T\bar{\kappa}_{xx} = \langle Q_x|W^{-1}|Q_x\rangle \approx \frac{\nu}{2}\frac{\pi^2 T^2 v_F^2}{3}\left(\frac{1}{\Gamma+\gamma} + \frac{\pi^2 T^2 \gamma}{3\Gamma(\Gamma+\gamma)p_F^2 v_F^2}\right)$$

$$+ \frac{2\pi^4 T^4}{2\, 3^2 p_F^2 \Gamma} + \frac{2\pi^4 T^4 \gamma}{9\Gamma(\Gamma+\gamma)p_F^2}$$

we only keep terms of order T^4, not $T^4\frac{\Gamma}{\gamma}$

$$\approx \frac{\nu}{2}\left[\frac{\pi^2 T^2 v_F^2}{3(\Gamma+\gamma)} + \frac{4\pi^4 T^4}{9 p_F^2 \Gamma}\right]$$

$$T\alpha_{xx} = \langle J_x|W^{-1}|Q_x\rangle = -e\frac{\nu}{2}\left[\cancel{\frac{\pi^2 T^2 v_F}{3p_F\Gamma}} - \frac{\pi^2 T^2 v_F}{3}\frac{\Gamma' - \gamma/p_F}{\Gamma(\Gamma+\gamma)}\right.$$

$$\left. + \frac{\pi^2 T^2}{3}v_F'\frac{\Gamma + \gamma\pi^2 T^2/3p_F^2 v_F^2}{\Gamma(\Gamma+\gamma)}\right]$$

$$\approx -\frac{e\nu}{2}\frac{\pi^2 T^2}{3}\left[\frac{v_F}{p_F(\Gamma+\gamma)} + \frac{2\gamma v_F}{p_F\Gamma(\Gamma+\gamma)} + \frac{v_F'}{\Gamma+\gamma} - \frac{v_F\Gamma'}{\Gamma(\Gamma+\gamma)}\right]$$

To understand these equations we first analyze them in the limit of negligible e-e scattering...

$\gamma \to 0$;

$$\sigma = \frac{\nu}{2}\frac{e^2 v_F^2}{\Gamma}.$$

$$\bar{\kappa} = \frac{\nu}{2}\frac{\pi^2 T^2 v_F^2}{3\Gamma} = \frac{\pi^2 T^2}{3e^2}\sigma = \bar{\kappa}_{\text{nonint}} \quad \text{Wiedemann-Franz Law!}$$

$$\alpha = -e\frac{\nu}{2}\frac{\pi^2 T}{3\Gamma}\left[\frac{v_F}{p_F} + v_F' - \frac{\Gamma' v_F}{\Gamma}\right]$$

Since
$$\frac{\partial\sigma}{\partial\mu} = \frac{1}{v_F}\frac{\partial\sigma}{\partial p_F} = \frac{1}{4\pi\hbar^2 v_F}\frac{\partial}{\partial p_F}\left(\frac{p_F v_F}{\Gamma}\right)$$
$$= \frac{\nu}{2p_F}\left(\frac{v_F}{\Gamma} + \frac{p_F v_F'}{\Gamma} - \frac{\Gamma'}{\Gamma^2}p_F v_F\right),$$

$$\alpha = -\frac{e\pi^2 T}{3}\frac{\partial\sigma}{\partial\mu} = \alpha_{\text{nonint}} \quad \text{Mott's Law!}$$

Now let us turn to the limit where e-e scattering is extremely large

$$\sigma = \frac{\nu}{2}\frac{e^2 v_F^2}{\Gamma}$$

$$\bar{\kappa} = \frac{\nu T}{2}\left(\frac{2\pi^2 T}{3p_F}\right)^2\frac{1}{\Gamma}$$

All transport coefficients controlled by momentum relaxation alone

$$\alpha = -e\nu\frac{\pi^2 T v_F}{3p_F\Gamma}$$

To understand this result, let us consider the following cartoon model, attempting to generalize the Drude model to thermoelectric transport

$$\underbrace{\Gamma P}_{\text{momentum relaxation rate}} = \underbrace{-enE}_{\text{Lorentz force}} \underbrace{- s\nabla T}_{\text{thermal analogue}} \leftarrow -\nabla \overset{\text{pressure}}{P} = -\rho\nabla\mu - s\nabla T$$

and $\nabla_\mu \to -E$, $\rho = -en\ldots$

$$J = -en\overset{\text{velocity of carriers?}}{v} \qquad P = mnv$$

$$Q = Tsv$$

these relations most naturally associated with a fluid

In a Fermi liquid: $s \simeq \frac{\pi^2 T}{3} \times \nu$, $m \simeq \frac{p_F}{v_F}$.

We predict $\sigma = \frac{e^2 n^2}{mn\Gamma}$, $\alpha = -\frac{ens}{mn\Gamma}$, $\bar{\kappa} = \frac{Ts^2}{mn\Gamma}$.

Recall that $n \sim p_F v_F \frac{\nu}{2}$, so we estimate:

$$\sigma = \frac{e^2}{\Gamma}\frac{n}{m} = \frac{e^2}{\Gamma}\frac{\nu}{2}v_F^2 \quad \checkmark \qquad = \sigma_{int}$$

$$\alpha = -\frac{e}{\Gamma}\frac{v_F}{p_F}\frac{\pi^2 T}{3}\nu \quad \checkmark \qquad = \alpha_{int}$$

$$\bar{\kappa} = \frac{T}{\Gamma}\left(\frac{\pi^2 T\nu}{3}\right)^2 \frac{1}{\frac{p_F}{v_F} p_F v_F \frac{\nu}{2}} = \frac{\nu T}{2\Gamma}\left(\frac{2\pi^2 T}{3 p_F}\right)^2 \quad \checkmark = \bar{\kappa}_{int}$$

To summarize...

$$\sigma \approx \frac{e^2 \nu v_F^2}{2\Gamma}, \quad \alpha = \frac{\Gamma}{\Gamma+\gamma}\alpha_{nonint} + \frac{\gamma}{\Gamma+\gamma}\alpha_{int}, \quad \bar{\kappa} = \frac{\Gamma}{\Gamma+\gamma}\bar{\kappa}_{nonint} + \frac{\gamma}{\Gamma+\gamma}\bar{\kappa}_{int}$$

The Wiedemann-Franz and Mott laws thus have a simple breakdown in the presence of momentum conserving interactions. We will later call this interaction dominated regime “hydrodynamic transport”

One more useful thing is to determine the experimentalist's thermal conductivity:

$$\kappa = \bar{\kappa} - \frac{T\alpha^2}{\sigma} \approx \frac{\pi^2 T}{3} \frac{\nu}{2} \frac{v_F^2}{\Gamma+\gamma} = \frac{\pi^2 T}{3e^2} \sigma \frac{\Gamma}{\Gamma+\gamma}$$

Lorenz ratio $\boxed{\frac{\mathcal{L}}{\mathcal{L}_0} = \frac{\Gamma}{\Gamma+\gamma}}$

Momentum conserving interactions suppress the experimentalist's thermal conductivity while they do not strongly affect the electrical conductivity. This is a key result. Especially at low temperatures this is a compelling transport signature for strong electron interactions and has been verified recently in a number of different compounds. But I caution that the story can be different for more complicated Fermi surfaces, and with umklapp etc...

2.8) Phonon-impurity scattering

Now we turn to the kinetic theory of thermal transport for phonons. Things are rather similar so I will try to not belabor the points so much. If the phonons are non-interacting and scattering off of impurities, then the same form of streaming and collision operators hold as for electrons, and all that changes is equilibrium distribution function...

phonon dispersion!

$$f = f_B(\omega(\vec{p})) - \frac{\partial f_{eq}}{\partial \varepsilon}\Phi + \cdots, \quad f_B(\omega) = \frac{1}{e^{\beta\omega(\vec{p})}-1}$$

no chemical potential because phonon number is NOT conserved.

$$\partial_t f + v\cdot\nabla_x f - \frac{\nabla T}{T}\cdot\varepsilon\vec{v}\left(-\frac{\partial f}{\partial \varepsilon}\right) = -W[f]$$

dc transport...: $|Q_i\rangle = \int d^d p \; \varepsilon(\vec{p}) v_i(\vec{p})$

$$-\frac{\nabla_i T}{T}|Q_i\rangle = W|\Phi\rangle, \quad T\bar{\kappa}_{ij} = \langle Q_i|W^{-1}|Q_j\rangle$$

In the presence of impurities, W is given by the same formula! Just need to change the equilibrium distribution in the inner product. Let's assume the disorder is relatively short range and "homogeneous" analogous to our discussion of electronic transport.

$$\langle Q_i|W^{-1}|Q_j\rangle = \int \frac{d^dp}{(2\pi\hbar)^d}\left(-\frac{\partial f_B}{\partial \varepsilon}\right)\varepsilon(p)^2\, v_i(p)\,v_j(p)\,\tau_{imp}(\varepsilon(p))$$

For bosons: $-\dfrac{\partial f_B}{\partial \varepsilon} = \dfrac{\beta e^{\beta\varepsilon}}{(e^{\beta\varepsilon}-1)^2} = \beta f_B(1+f_B)$.

For short range disorder:

$$\langle Q_i|W^{-1}|Q_j\rangle = \tau_{imp}\int \frac{d^dp}{(2\pi\hbar)^d}\,\frac{\varepsilon(p)^2}{T}\,v_i(p)\,v_j(p)\,f_B(1+f_B) = T\bar{\kappa}_{ij}$$

Isotropic acoustic phonons: $\varepsilon(p)\approx v_p|\vec{p}|$, $v_i \propto p_i$:

$$\bar{\kappa} = \frac{\tau_{imp}}{T^2}\int_0^\infty \frac{d\varepsilon}{(2\pi\hbar)^d}\,\overset{\text{constant}}{\Omega_d}\,\frac{\varepsilon^{d-1}\varepsilon^2}{v_p^d}\,\overset{v_iv_j}{\frac{v_p^2}{d}}\,f_B(1+f_B)$$

$$f_{eq}\approx\begin{cases}1/\beta\varepsilon & \beta\varepsilon\ll 1\\ e^{-\beta\varepsilon} & \beta\varepsilon\gg 1\end{cases}\;:\quad \bar{\kappa}\propto\left[\frac{T}{\hbar v_p}\right]^d\tau_{imp}\,v_p^2\times\underbrace{\int_0^\infty\frac{dx\;x^{d+1}\,\Omega_d}{d(2\pi)^d}\,\frac{e^x}{(e^x-1)^2}}_{\text{finite constant.}}$$

Hence short range impurity resistance from acoustic phonons:

$$\bar{\kappa}_{ph}\sim T^d \sim \#\text{ of thermally excited phonons}$$

From optical phonons where $\varepsilon(p)\approx\omega_0$:

$$\bar{\kappa}\sim\frac{\tau_{imp}}{T^2}\,\omega_0^2\left[\int\frac{d^dp\;v^2}{(2\pi\hbar)^d}\right]f_B(\omega_0)(1+f_B(\omega_0))\sim\begin{cases}T^0 & T\gg\omega_0\\ e^{-\omega_0/T} & T\ll\omega_0\end{cases}$$

2.9) Phonon-phonon scattering

We continue our discussion of phonon contribution to thermal conductivity, now turning to the possibility of phonon umklapp scattering.

$$H=\sum_{\alpha,q}\omega_{q\alpha}a^{\dagger}_{q\alpha}a_{q\alpha}+\underbrace{\sum U_{\alpha\beta\gamma}(q,k)\,a^{\dagger}_{(q+k)\alpha}a_{q\beta}a_{k\gamma}+H.c.}_{1\leftrightarrow 2\text{ phonon process, arising from }3^{rd}\text{ order corrections in the potential energy of crystal}}+\cdots$$

For simplicity let's just go directly to our variational estimate:

$$\langle\Phi|W|\Phi\rangle=\beta\int\frac{d^dp\,d^dp'}{(2\pi\hbar)^{2d}}f_B(\vec{p})f_B(\vec{p}\,')(1+f_B(\vec{p}+\vec{p}\,'))\left(\Phi(\vec{p})+\Phi(\vec{p}\,')-\Phi(\vec{p}+\vec{p}\,')\right)^2$$
$$\times|U|^2$$

Where the derivation of this result is similar to that for electron electron scattering.

For phonon scattering one expects $|U(q_1,q_2,q_3)|^2\sim|q_1||q_2||q_3|$

At high temperatures, $f_B(q)\sim\frac{T}{\omega(q)}$. For a "typical" Φ

$$\langle\Phi|W|\Phi\rangle\sim\beta\int\frac{d^dq_1\,d^dq_2\,d^dq_3}{(2\pi\hbar)^{2d}}\delta(q_3-q_1-q_2)\prod_{i=1}^{3}\frac{|q_i|T}{\omega(q_i)}\left(\Phi(q_3)-\Phi(q_1)-\Phi(q_2)\right)^2$$

Assuming $\omega\sim v_{ph}q$ for acoustic modes & $\omega\sim\omega_0+\cdots$ for optical modes:

$$\langle\Phi|W|\Phi\rangle\sim T^2$$

Set $\Phi(\vec{p})\sim\vec{p}$

Just like with electrons, only umklapp scattering relevant

Now we evaluate

$$\langle Q_x | P_x \rangle = \int \frac{d^d p}{(2\pi\hbar)^d} \left(-\frac{\partial f_B}{\partial \varepsilon}\right) \varepsilon v_x p_x = \int \frac{d^d p}{(2\pi\hbar)^d} \left(-\frac{\partial f_B}{\partial p_x}\right) \varepsilon p_x$$

$$= \int \frac{d^d p}{(2\pi\hbar)^d} f_B (\varepsilon + p_x v_x)$$

using fact that BZ has no boundary, & integrate by parts

Now note that for bosons, $f_B = -T\frac{\partial}{\partial \varepsilon} \log(1-e^{-\varepsilon/T})$, so

$$\int \frac{d^d p}{(2\pi\hbar)^d} \left(-T \frac{\partial}{\partial \varepsilon} \log(1-e^{-\varepsilon/T})\right) p_x \frac{\partial \varepsilon}{\partial p_x} = \int \frac{d^d p}{(2\pi\hbar)^d} \left(-T \log(1-e^{-\varepsilon/T})\right)$$

$$= P(T) \quad \text{(pressure!)}$$

Gibbs-Duhem identity $\varepsilon + P = \underbrace{\cancel{\mu n}}_{0 \text{ for phonons!}} + Ts$

$$\boxed{\langle Q_x | P_x \rangle = Ts}$$

This identity is general

At high T, entropy of phonons is $s \sim T^0$. Hence

$$\frac{1}{T\kappa} \lesssim \frac{\langle P_x | W | P_x \rangle}{\langle P_x | Q_x \rangle^2} \sim T^0, \quad \text{or} \quad \boxed{\kappa \sim \frac{1}{T}}$$

At low temperatures only umklapp is present, because there is no Fermi surface for bosonic phonons. So we expect

$$\kappa \sim e^{T_0/T} \quad \text{as} \quad T \to 0$$

Like for electrons, impurity scattering will dominate at low temperatures. At intermediate temperature scales there can be a very nasty interplay of different effects of umklapp, band structure etc. there is not a clean theory here

2.10) Electron-phonon scattering

In this part we now consider the electron phonon scattering integral.

$$\begin{pmatrix} |\Phi\rangle \\ |\phi\rangle \end{pmatrix} \begin{matrix} \leftarrow \text{electron distribution} \\ \leftarrow \text{phonon distribution} \end{matrix} = |\varphi\rangle \qquad |\Lambda|^2 \sim q$$

If $H = \sum \Lambda_{\vec{q}\vec{k}_1\vec{k}_2} a_{\vec{q}} c^\dagger_{\vec{k}_1} c_{\vec{k}_2} + \text{H.c.}$, then collision integral

$$\langle\varphi|W|\varphi\rangle = \pi\beta \int \frac{d^d\vec{q}\, d^d\vec{k}_1\, d^d\vec{k}_2}{(2\pi)^{3d}} \delta(\vec{q}+\vec{k}_2-\vec{k}_1)\delta(\varepsilon_{\vec{k}_2}+\omega_{\vec{q}}-\varepsilon_{\vec{k}_1})|\Lambda|^2 \times$$

$$f_B(\vec{q}) f_F(\vec{k}_2)(1-f_1(\vec{k}_1)) \left(\phi(\vec{q})+\Phi(\vec{k}_2)-\Phi(\vec{k}_1)\right)^2$$

Bose-Einstein dist.   Fermi-Dirac dist.

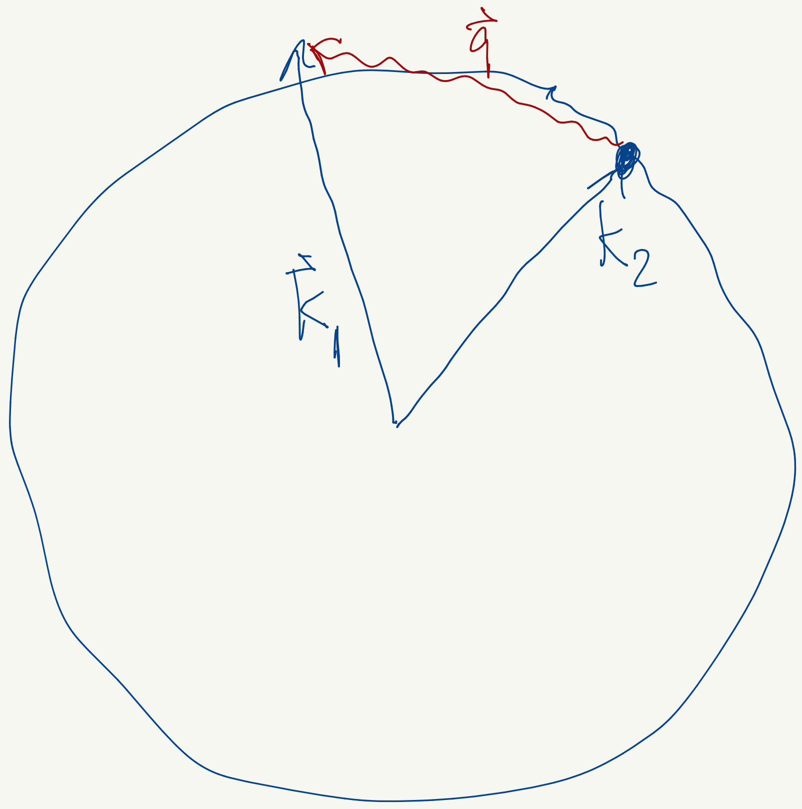
This result can be derived analogously to earlier results, and so we state it without proof.

We now make “Bloch’s ansatz” which is that the phonon distribution is in equilibrium. We will return to this assumption I a bit. So $\phi(\vec{q}) = 0$ in our variational estimate.

y about the acoustic phonons. Another reasonable assumption is that the phonon energy
um is very small compared to the electron energy, since in a typical metal

$$\omega_q \sim v_{ph}|q|, \qquad \varepsilon_k \sim v_F(|\vec{k}|-k_F)$$

$$v_{ph} \sim 10^3 \text{ m/s}, \qquad v_F \sim 10^5 \text{ m/s}$$



Electrons will stay close to the Fermi surface

Integrate out $\vec{k}_1$:

$$\langle \Phi | W | \Phi \rangle = \pi \int \frac{d^d \vec{k}_2 d^d \vec{q}}{(2\pi)^{3d}} \delta(\varepsilon_{\vec{k}_2} + \omega_{\vec{q}} - \varepsilon_{\vec{q}+\vec{k}_2}) \beta |\Delta|^2 f_B(\vec{q}) f_F(\vec{k}_2)(1 - f_F(\vec{k}_2+\vec{q}))(\Phi(\vec{k}_2) - \Phi(\vec{k}_2+\vec{q}))^2$$

Write $\int d^d \vec{k}_2 = \int \frac{d\varepsilon_2}{|v(\vec{k}_2)|} \int_{FS} dk_{\parallel 2}$. If $\Phi \approx \varepsilon$-independent...

$$\langle \Phi | W | \Phi \rangle \sim \pi \int \frac{d^{d-1} k_{\parallel 2} d^d \vec{q}}{(2\pi)^{2d}} |\Delta|^2 f_B(\vec{q}) (\Phi(\vec{k}_2) - \Phi(\vec{k}_2+\vec{q}))^2 \times \underbrace{\Theta(T - \omega_{\vec{q}})}_{\text{from integral over } \beta f_F(1-f_F)}$$

At low temperatures: using $f_B |\Delta|^2 \sim \begin{cases} T \times q^0 & \omega_q \ll T \\ e^{-\beta \omega_q} & \omega_q \gg T \end{cases}$

$$\langle \Phi | W | \Phi \rangle \sim \int_0^{T/v_{ph}} \frac{d^d q}{(2\pi)^d} \int \frac{d^{d-1} k_\parallel}{(2\pi)^d} f_B\left(\frac{v_{ph} q}{T}\right) (\Phi(\vec{k}_2) - \Phi(\vec{k}_2 + \vec{q}))^2$$

For generic perturbation Φ... $\langle \Phi | W | \Phi \rangle \sim T^d$

— single particle relaxation time!

The transport coefficient however comes from ansatz $\Phi(\vec{k}) = k_x$ .... $|P_x\rangle$

$$\langle P_x | W | P_x \rangle \sim \int_0^{T/v_{ph}} \frac{d^d q}{(2\pi)^d} \int \frac{d^{d-1} k_\parallel}{(2\pi)^d} f_B\left(\frac{v_{ph} q}{T}\right) \vec{q}^{\,2} \sim \underline{\underline{T^{d+2}}}$$

Since $\langle P_x | J_x \rangle = -e \int \frac{d^d \vec{k}}{(2\pi)^d} \left(-\frac{\partial f_F}{\partial \varepsilon}\right) \frac{\partial \varepsilon}{\partial} = -e \int \frac{d^d \vec{k}}{(2\pi)^d} f_F = -en \sim T^0$,

$$\rho_{xx} \lesssim \frac{\langle P_x | W | P_x \rangle}{\langle P_x | J_x \rangle^2} \sim T^{d+2}$$

(T^5 for conventional metal)

Surprisingly, that T^5 is not easy to observe in an actual metal. The problem is that you have to be at extremely low temperatures where impurity scattering dominates. Because at higher temperatures...

Once $k_F \sim T/v_{ph}$:

Fermi wave vector/momentum

$$\int_0^{k_F} \frac{d^d q}{(2\pi)^d} \int_{FS} \frac{d^{d-1} k_{\parallel}}{(2\pi)^d} \vec{q}^{\,2} f_B\left(\frac{v_{ph} q}{T}\right)$$

$$\sim \int_0^{k_F} \frac{d^d q}{(2\pi)^d v_{ph} q} \int_{FS} \frac{d^{d-1} k_{\parallel}}{(2\pi)^d} T \vec{q}^{\,2} \sim T$$

Bloch-Grüneisen temperature

Hence when $T \gtrsim v_{ph} k_F \equiv T_{BG}$, $\boxed{\rho \sim T}$

Estimate for a "usual" metal: $T_{BG} \sim \frac{\hbar}{k_B} k_F v_{ph}$

$$\frac{T_{BG}}{1\ K} \sim \frac{10^{-34}}{10^{-23}} 10^{10} 10^{3} \sim 100\ K$$

By room temperature the phonon induced resistivity of most metals will be linear in temperature.

Lastly let's briefly mention optical phonon contributions. These are not usually so relevant for real metals

$\omega_q \approx \omega_0$, so

$$\langle \Phi | W | \Phi \rangle \sim e^{-\omega_0/T} \quad \text{if } T \ll \omega_0$$

$$\langle \Phi | W | \Phi_0 \rangle \sim \int_0^{k_F} \frac{d^d q}{(2\pi)^d} \int_{FS} \frac{d^{d-1} k_{\parallel}}{(2\pi)^d} \vec{q}^{\,2} f_B\left(\frac{\omega_0}{T}\right) \quad \text{if } \omega_0 \ll T \ll T_F$$

Hence $\rho \sim T$

2.11) Phonon drag

Now let us relax Bloch's assumption that the phonons are in thermal equilibrium. In general the problem simply becomes a more involved variational one, etc...so let's focus on a simple example where we have electron-impurity scattering, momentum conserving electron phonon scattering, and momentum relaxing phonon scattering

$$\langle \varphi | W | \varphi \rangle = \langle \Phi | W_{ei} | \Phi \rangle + \langle \phi | W_p | \phi \rangle + \langle \varphi | W_{ep} | \varphi \rangle$$

$$= \int \frac{d^d \vec{k}_1 d^d \vec{k}_2}{(2\pi)^{2d}} \left(-\frac{\partial f_F}{\partial \varepsilon} \right) \delta(\varepsilon_{\vec{k}_1} - \varepsilon_{\vec{k}_2}) |V|^2 (\Phi(\vec{k}_1) - \Phi(\vec{k}_2))^2 + \int \frac{d^d \vec{q}_1 d^d \vec{q}_2}{(2\pi)^{2d}} \left(-\frac{\partial f_B}{\partial \varepsilon} \right) (\phi(\vec{q}_1) - \phi(\vec{q}_2))^2 \ldots$$

$$+ \int \frac{d^d \vec{k}_1 d^d \vec{k}_2 d^d \vec{q}}{(2\pi)^{3d}} |\Delta|^2 \delta(\vec{k}_1 - \vec{k}_2 - \vec{q}) \delta(\varepsilon_{\vec{k}_1} - \varepsilon_{\vec{k}_2} - \omega_{\vec{q}}) f_B(\vec{q}) f_F(\vec{k}_2)(1 - f_F(\vec{k}_1)) (\Phi(\vec{k}_1) - \Phi(\vec{k}_2) - \phi(\vec{q}))^2$$

Trial function: $\Phi(\vec{k}) = k_x$, $\phi(\vec{q}) = q_x \times \alpha$;

$$\langle \varphi | W | \varphi \rangle = \langle P_x^e | W_{ei} | P_x^e \rangle + \langle P_x^p | W_p | P_x^p \rangle \alpha^2$$

$$+ \underbrace{\int \frac{d^d \vec{k}_1 d^d \vec{k}_2 d^d \vec{q}}{(2\pi)^{3d}} |\Delta|^2 (1-\alpha)^2 q^2 \times \ldots\ldots}$$

$$= (1-\alpha^2) \langle P_x^e | W_{eph} | P_x^e \rangle$$

calculation of previous section!

$$\rho_{xx} \leq \min_\alpha \frac{\langle \varphi | W | \varphi \rangle}{\langle \varphi | J_x \rangle^2} = \min_\alpha \frac{\langle P_x | W_{ei} | P_x \rangle + \alpha^2 \langle P_x | W_{ph} | P_x \rangle + (1-\alpha)^2 \langle P_x^e | W_{eph} | P_x^e \rangle}{\langle P_x | J_x \rangle^2}$$

$$= \rho_{ei} + \rho_{Bloch} \frac{\langle P_x | W_{ph} | P_x \rangle}{\langle P_x^e | W_{eph} | P_x^e \rangle + \langle P_x | W_{ph} | P_x \rangle}$$

The Bloch resistance is thus reduced by the relative momentum relaxation rate of phonons to umklapp or impurities, relative to the momentum conserving rates, weighed by number of excitations involved in each....

This can get rather messy, and in many metals Bloch's approximation is reasonable. But at very low temperatures it may be the case that phonon umklapp and impurity scattering are both extremely suppressed in which case phonon scattering simply drops out of the expression for electrical resistance!