Physics 7450, Fall 2019

3. Hydrodynamics

3.1) Diffusion on the Fermi surface

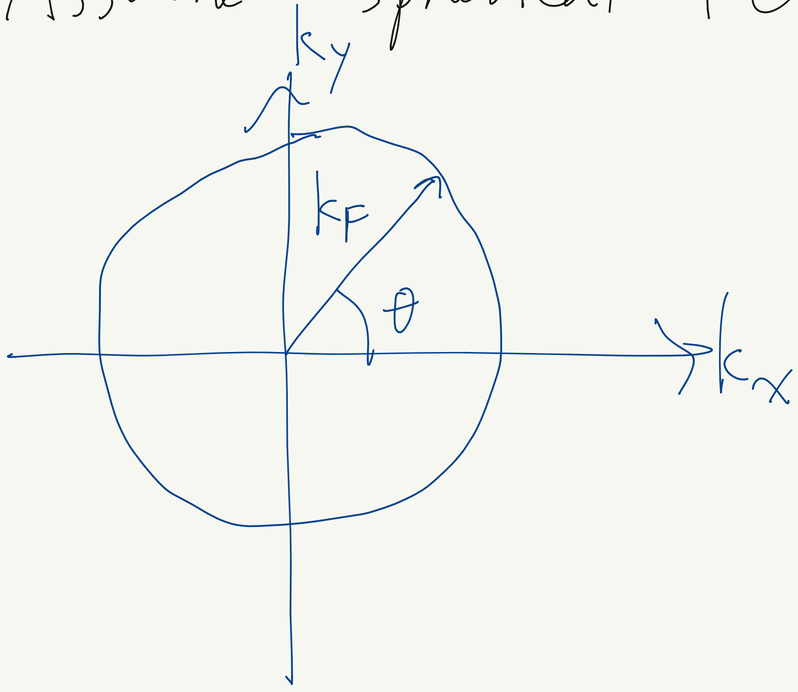
So far we have essentially been assuming that the system is relaxing to equilibrium in a spatially homogeneous way. It is now time to relax that assumption. Recall that in our study of short range disorder in a low temperature Fermi liquid, we found that

$$\langle p|W|\Phi\rangle = \frac{1}{\tau_{\mathrm{imp}}}\left[\Phi(\vec{p}) - \int_{FS}\frac{d^{d-1}p'}{\mathrm{Vol(FS)}}\Phi(\vec{p}\,')\right]\left(-\frac{\partial f}{\partial \varepsilon}\right)_{\vec{p}}$$

outgoing

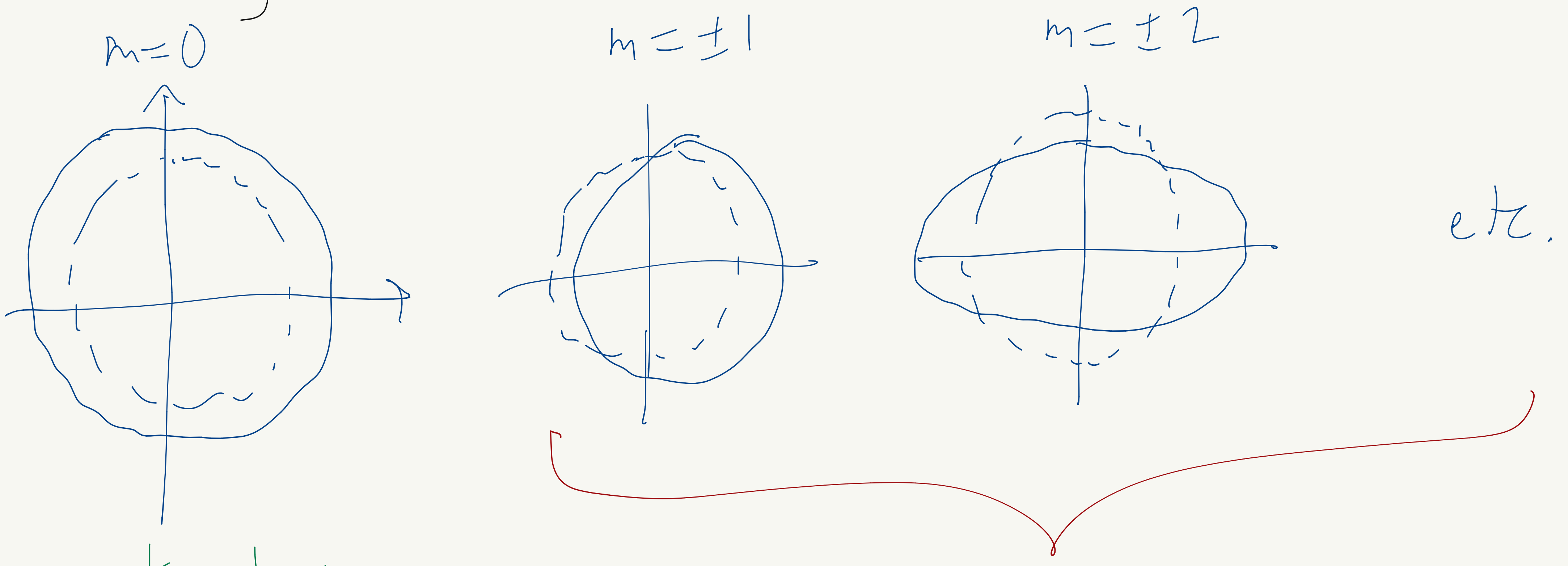
incoming, equally likely from any point....

Work in d=2. Assume spherical Fermi surface



Consider basis

$$|m\rangle = \int d^dp \;\; e^{-im\theta}|\vec{p}\rangle$$



Cannot relax: charge conservation

Can relax due to impurity scattering...

$$\langle m'|W|m\rangle = \frac{\nu}{2\pi}\int d\theta\, e^{-im'\theta}\frac{1}{\tau_{imp}}\left[e^{im\theta} - \int_0^{2\pi}\frac{d\theta}{2\pi}e^{im\theta}\right]$$

$$= \frac{\nu}{2\pi}(1-\delta_{m0})\delta_{m,m'}\frac{1}{\tau_{imp}}\,.$$

$$\frac{e^{i\theta}-e^{-i\theta}}{2i}$$

Now we need to turn on the “streaming” terms in the kinetic equation. These become non trivial if we have any inhomogeneity.

$$\langle m'|v_x\partial_x + v_y\partial_y|m\rangle \approx \frac{\nu}{2\pi}\int_0^{2\pi}d\theta\, e^{i(m-m')\theta}\underbrace{(\cos\theta\,\partial_x + \sin\theta\,\partial_y)}_{\frac{e^{i\theta}}{2}(\partial_x - i\partial_y) + \frac{e^{-i\theta}}{2}(\partial_x + i\partial_y)}v_F$$

$$= \frac{\nu v_F}{2}\left[\delta_{m',m+1}\underbrace{(\partial_x - i\partial_y)}_{\partial_+} + \delta_{m',m-1}\underbrace{(\partial_x + i\partial_y)}_{\partial_-}\right]$$

So let’s write down the kinetic equations, restricted to these Fermi surface fluctuations...

$$|\Phi\rangle = \sum_{m\in\mathbb{Z}}\Phi_m(x,t)|m\rangle$$

$$\partial_t\Phi_0 + \frac{v_F}{2}\left[\partial_+\Phi_{-1} + \partial_-\Phi_1\right] = 0$$

$$\partial_t\Phi_1 + \frac{v_F}{2}\left[\partial_+\Phi_0 + \partial_-\Phi_2\right] = -\frac{1}{\tau}\Phi_1$$

etc...

On time scales short compared to the impurity scattering time, we can reduce the problem to non interacting “ballistic physics”. This regime will be discussed more in the next part of the course. So what happens on long time scales?

Claim: $$\frac{v_F}{2}\left[\partial_-\Phi_0 + \partial_+\Phi_2\right] \approx -\frac{\Phi_1}{\tau}.$$

For suppose we have the equation

$$\partial_t \varphi + \frac{\varphi}{\tau} = f(t) \leftarrow \text{"given" source.}$$

$$\partial_t\left(\varphi e^{t/\tau}\right) = f e^{t/\tau}$$

$$\varphi = \int_{-\infty}^{t} dt'\, f(t')\, e^{-(t-t')/\tau}$$

If f varies slowly on time scale τ:

$$\varphi(t) = \int_{-\infty}^{t} dt' \left[f(t) - f'(t)(t-t') + \cdots\right] e^{-(t-t')/\tau}$$

$$= \tau f(t) - \underbrace{\tau^2 f'(t)}_{\text{negligible by assumption!}} + \cdots$$

Hence we arrive at the following tower of equations...

$$\partial_t\Phi_0 + \frac{v_F}{2}\left[\partial_+\Phi_{-1} + \partial_-\Phi_1\right] = 0$$

$$\cancel{\partial_t\Phi_{\pm1}} + \frac{v_F}{2}\left[\partial_\pm\Phi_0 + \partial_\mp\Phi_{\pm2}\right] = -\frac{1}{\tau}\Phi_{\pm1}$$

$$\cancel{\partial_t\Phi_{\pm2}} + \frac{v_F}{2}\left[\partial_\pm\Phi_{\pm1} + \partial_\mp\Phi_{\pm3}\right] = -\frac{1}{\tau}\Phi_{\pm2} \qquad \cdots\cdots$$

If system varies on spatial length scale d....

$$\Phi_{\pm m} \sim \left(\frac{\tau v_F}{d}\right)^m \Phi_0 \,!$$

Hence $-\frac{1}{\tau}\Phi_{\pm 1} \approx \frac{v_F}{2}\partial_\pm \Phi_0$ and we conclude that

$$\partial_t \Phi_0 - \frac{v_F^2\tau}{2}\partial_-\partial_+\Phi_0 = 0 = \partial_t\Phi_0 - D(\partial_x^2+\partial_y^2)\Phi_0$$

$D=\frac{v_F^2\tau}{2}$ is a diffusion constant.

We interpret Phi_0 as the charge density, since after all...

$$\langle 0|\Phi\rangle = \int\frac{d^2\vec{p}}{(2\pi\hbar)^2}\Phi(\vec{p})\,\delta(\varepsilon(\vec{p})-\mu) = n - n_{eq}.$$

$$\boxed{\partial_t n = D\nabla^2 n}$$

This is the simplest hydrodynamic equation. As we make precise later, hydrodynamics is the effective theory of how conserved quantities evolve on long time scales.

Let's go ahead and think about the conductivity of this theory. Since

$$|J_x\rangle = -ev_F\frac{(|1\rangle+|-1\rangle)}{2},$$

$$\sigma_{xx} = \sigma = \langle J_x|W^{-1}|J_x\rangle = \frac{e^2v_F^2\nu\tau}{2} = \nu e^2 D$$

Since $\frac{\partial \overset{\text{charge density}}{\rho}}{\partial\mu_{ch}} = \nu\frac{\partial(-en)}{\partial\left(\frac{\mu}{-e}\right)} = \nu e^2 = \chi \leftarrow$ Charge susceptibility,

Not accidental that $\chi = e^2\langle 0|0\rangle$: $|0\rangle$ denotes charge fluctuations, and μ is conjugate to ρ thermodynamically.

Thus we derived an Einstein relation

$$\underline{\sigma = \chi D}$$

3.2) Hydrodynamics as an effective theory

There are a few shortcomings in our discussion thus far. We have derived some equations governing the dynamics of globally conserved quantities, but they're all within linear response. Also, why did these equations take the form that they did? We'll now give a second perspective which should help answer these questions.

Suppose that we have a system with a known set of conserved quantities. Our postulate is that on long time scales, the only slow degrees of freedom are the conserved quantities themselves...

Let $\rho^A(x)$ denote conserved quantities:

$$\frac{d}{dt}\int d^dx\, \rho^A(x) = 0.$$

Assume locality, which implies that:

$$\frac{\partial}{\partial t}\rho^A(x) + \nabla\cdot\vec{J}^A(x) = 0 \quad \text{for some } \vec{J}^A$$

Our postulate can be stated more formally as...

$\vec{J}^A$ is a function of only $\rho^A(x)$ <u>OR</u> its x & t derivatives

This conservation law should hold quantum mechanically (as an operator statement) as we will discuss near the end of the course. For now let's treat this as a classical constraint.

We will now proceed by carrying out a Taylor expansion of the currents, order by order in derivatives. This is called a derivative expansion or gradient expansion.

<u>effective theory approach</u>

As a simple example, let’s consider our theory of charge diffusion along the Fermi surface from before.

Only conserved quantity is ρ, charge density. What can $\vec{J}$ be?

$$J_i = \bar{J}_i(\rho) + b_i(\rho)\partial_t \rho - D_{ij}(\rho)\partial_j \rho + \tilde{b}_i(\rho)\partial_t^2 \rho + \cdots ?$$

If Spatial isotropy: all coefficients above $\sim \delta_{ij}$ [or ϵ_{ijk}?]

Up to first derivative order:

$$J_i = -D(\rho)\partial_i \rho + \cdots$$

$$\partial_t \rho + \nabla \cdot J = 0 \Longrightarrow \boxed{\partial_t \rho = \nabla \cdot (D(\rho)\nabla \rho)}$$

This diffusion equation is the simplest hydrodynamic theory, describing the relaxation of charge.

There is one more important ingredient to hydrodynamics, corresponding to the second law of thermodynamics.

$$\partial_t \int d^d x \underset{\substack{\uparrow \\ \text{entropy density}}}{s} \geq 0 \ ?$$

Assume $s = s(\rho)$.

$$\partial_t \int d^d x\, s = \int d^d x \frac{\partial s}{\partial \rho}\partial_t \rho = \int d^d x \frac{\partial s}{\partial \rho}\left(\nabla \cdot (D\nabla \rho)\right)$$

$$= \int d^d x\, D(\nabla \rho)^2 \underbrace{\left(-\frac{\partial^2 s}{\partial \rho^2}\right)}_{}$$

<0 in a generic thermodynamically <u>stable</u> phase

Conclusion: $\boxed{D \geq 0}$

As we will see, also necessary for diffusion to be well behaved.

It is straightforward to generalize these arguments to a theory with multiple conserved quantities that are scalars under rotation.

$$\partial_t \rho^A + \nabla\cdot J^A = 0, \qquad J^A = -D^{AB}\nabla\rho^B .$$

D^{AB} is a positive definite matrix,

A classic example of this would be the diffusion of charge and energy in a metal (in the absence of long range Coulomb interactions, which we will shortly get to...)

$$\partial_t \begin{pmatrix}\rho\\ \varepsilon\end{pmatrix} = \underset{\text{matrices}}{D}\,\nabla^2\begin{pmatrix}\rho\\ \varepsilon\end{pmatrix}$$

Since $\begin{pmatrix}\delta\rho\\ \delta\varepsilon\end{pmatrix} = \chi\begin{pmatrix}\delta\mu\\ \delta T\end{pmatrix}$, D can be related to $\begin{pmatrix}\sigma & T\alpha\\ T\alpha & T\bar{\kappa}\end{pmatrix}$,

but the precise formulas are not enlightening.

Reference: 1405.3651

Finally, let's discuss a system with a conserved energy, charge and momentum — just like a classical gas. Since momentum is now a vector under rotations, even in an isotropic fluid the currents become a little more interesting.

$$\underset{\text{densities}}{\begin{pmatrix}\rho\\ \varepsilon\\ P_i\end{pmatrix}} \longrightarrow \underset{\text{currents}}{\begin{pmatrix}J\\ J_E\\ \tau_{ij}\end{pmatrix}}$$

$$\partial_t\rho + \partial_i J_i = 0$$
$$\partial_t\varepsilon + \partial_i J_{Ei} = 0$$
$$\partial_t P_i + \partial_j \tau_{ij} = 0$$

$$\begin{pmatrix}J_i\\ J_{Ei}\end{pmatrix} = \overset{\text{matrix}}{A}(\rho,\varepsilon,P^2)\,\partial_i\begin{pmatrix}\rho\\ \varepsilon\end{pmatrix} + \begin{pmatrix}b_{01}\\ b_{02}\end{pmatrix}P_i + \begin{pmatrix}b_{11}\\ b_{12}\end{pmatrix}\partial_t P_i + \cdots$$

$$\tau_{ij} = \delta_{ij} f(\partial_k P_k, \rho, \varepsilon, \ldots) + g_1\partial_i P_j + g_2\partial_j P_i + \cdots$$

But not all of these coefficients need to show up. There are two things to deal with: symmetry forbidden terms, and terms which correspond to a change in fluid frame. Let's start with the first.

Suppose we have rotational invariance. Then angular momentum is conserved:

$$\partial_t\left[x_i P_j - x_j P_i\right] + \partial_k J^{\text{ang}}_{ijk} = 0 ?$$

$$\partial_t\left[x_i P_j - x_j P_i\right] = -x_i \partial_k \tau_{jk} + x_j \partial_k \tau_{ik}$$

$$= -\partial_k\left[x_i \tau_{jk} - x_j \tau_{ik}\right] + \underbrace{\left[\tau_{ij} - \tau_{ji}\right]}_{=0!}$$

Thus, $g_1 = g_2$.

Now we come to the question of fluid frame

$$\tau_{ij} = a_0 \delta_{ij} \partial_t \varepsilon + a_2 \delta_{ij} \partial_k P_k + \cdots$$

If $\partial_t \varepsilon = a_1 \partial_k P_k + \cdots$, then

$$\underbrace{\tau_{ij} = (a_2 + a_0 a_1)\delta_{ij} \partial_k P_k + \cdots}$$

this is an <u>equivalent hydrodynamics</u>, w/ identical physical predictions.

So we would like to only keep track of terms in the currents which are physically distinguishable. This is called “fixing the fluid frame” in hydrodynamics. A common choice for nonrelativistic theories is to set all terms involving time derivatives to zero.

One final thing that is often done is to replace densities of conserved quantities with their thermodynamic conjugates

charge density $\rho \longrightarrow$ chemical potential μ
energy density $\varepsilon \longrightarrow$ temperature T (schematic...)
momentum $P_i \longrightarrow$ <u>fluid</u> velocity u_i

$$\begin{pmatrix} J_i \\ J_{Ei} \end{pmatrix} = \begin{pmatrix} n_c \\ n_E \end{pmatrix} u_i - \underbrace{\Sigma_0}_{\text{"incoherent" conductivity matrix}} \partial_i \begin{pmatrix} \mu \\ T \end{pmatrix} + \dots$$

$$\tau_{ij} = -\underbrace{\eta}_{\text{shear viscosity}} \left[\partial_i u_j + \partial_j u_i - \frac{2}{d} \delta_{ij} \partial_k u_k \right] - \underbrace{\zeta}_{\text{bulk viscosity}} \delta_{ij} \partial_k u_k + \delta_{ij} A$$

The constants A, n_c, n_E are NOT arbitrary. In the limit of small velocity, we will find that

$$A = P \text{ (thermodynamic pressure)}; \quad n_c = \overset{\text{charge density}}{\rho}, \quad n_E = \overset{\text{energy density}}{\varepsilon} + P$$

The procedure for deriving hydrodynamic equations more generally follows this procedure — using thermodynamic constraints to fix the leading order (zero derivative) coefficients in the currents, then fixing fluid frame, and finally using second law of thermodynamics. In this context, the result of the second law is that

$$\eta \geq 0, \quad \zeta \geq 0, \quad \Sigma_0 \text{ positive semidefinite}$$

There is one final point worth making. Independently of a kinetic theory description, namely purely from effective theory principles, we observe the following basic estimates of the derivative coefficients

dimensionless parameter governing effective theory of hydro:

$$\underbrace{\tau_{int}}_{\text{mean free time}} \partial_t \qquad \underbrace{l_{int}}_{\text{mean free path}} \partial_x$$

hydro is appropriate on times $\Delta t \gg \tau_{int}$

& lengths $\Delta x \gg l_{int}$

We will see explicitly how hydro breaks down in the next part of the course, on shorter length and time scales.

Estimate: $\eta, \zeta, \Sigma_0 \sim l_{int}$ or τ_{int}, as they come w/ extra derivatives

3.3) Hydrodynamics from a general kinetic theory

We now derive the linearized hydrodynamic equations from a generic kinetic theory, and hence explicitly calculating all the phenomenological coefficients of hydro. we focus on a theory with a conserved charge, energy and momentum, these techniques generalize to any kinetic theory.

Our first step is to find an explicit expression for all of the currents.

let $|\rho^A\rangle$ be the set of null vectors of W: $W|\rho^A\rangle = 0$.

$$\langle\rho^A|\left(\partial_t|\Phi\rangle + \vec{v}\cdot\nabla_x|\Phi\rangle\right) = -\langle\rho^A|W|\Phi\rangle = 0.$$

Defining $\rho^A(x,t) = \langle\rho^A|\Phi(x,t)\rangle$,

$$\vec{J}^A(x,t) = \langle\rho^A|\vec{v}|\Phi(\vec{x},t)\rangle = \langle J^A|\Phi(x,t)\rangle$$

Next, suppose we apply an infinitesimal change to the distribution function

$$f_{eq}\left(\frac{\varepsilon-\mu}{T}\right) \to f_{eq}\left(\frac{\varepsilon-\mu-\delta\mu-\overset{\text{fluid velocity}}{\delta\vec{u}}\cdot\vec{p}}{T+\delta T}\right)$$

$$f = f_{eq} - \frac{\partial f_{eq}}{\partial\varepsilon}\Phi, \quad |\Phi\rangle = \delta\mu|n\rangle + \left[\frac{\delta T}{T}|\varepsilon\rangle - \frac{\mu}{T}\delta T|n\rangle\right] + \delta\vec{u}\cdot|\vec{P}\rangle,$$

where the list of conserved quantities is

charge: $|n\rangle = \int d^dp\ |\vec{p}\rangle$

energy: $|\varepsilon\rangle = \int d^dp\ \varepsilon(\vec{p})|\vec{p}\rangle$

momentum: $|P_i\rangle = \int d^dp\ p_i|\vec{p}\rangle$

Part of our hydrodynamic ansatz (which we will carefully justify in general, and already saw in our simple examples of diffusion on the Fermi surface) is that on long time and length scales...

$$|\Phi(x,t)\rangle = \delta\mu(x,t)|n\rangle + \delta T(x,t)\left[\frac{|\varepsilon\rangle}{T} - \frac{\mu}{T}|n\rangle\right] + \delta u_i(x,t)|P_i\rangle . \quad \text{So}$$

$$\langle J_i|\Phi\rangle = \left(\delta\mu - \frac{\mu}{T}\delta T\right)\langle J_i|n\rangle + \frac{\delta T}{T}\langle J_i|\varepsilon\rangle + \delta u_j\langle J_i|P_j\rangle$$

With isotropy, though with any sort of “mirror” symmetries, only the last term above is non-trivial.

$$\langle J_i|P_j\rangle = \int\frac{d^dp}{(2\pi\hbar)^d}\left(-\frac{\partial f_{eq}}{\partial\varepsilon}\right)p_j v_i = \int\frac{d^dp}{(2\pi\hbar)^d}\left(-\frac{\partial f_{eq}}{\partial p_i}\right)p_j = \int\frac{d^dp}{(2\pi\hbar)^d}f_{eq}\delta_{ij} = \overset{\substack{\text{number}\\ \text{density!}\\ \downarrow}}{n}\,\delta_{ij}$$

Similar calculations lead to the following

$$\langle J^E_i|P_j\rangle = (\varepsilon+P)\delta_{ij}$$ We did this calculation back in **section 2.9**

$$\langle \tau_{ij}|n\rangle = \int\frac{d^dp}{(2\pi\hbar)^d}\left(-\frac{\partial f}{\partial\varepsilon}\right)p_i v_j = \int\frac{d^dp}{(2\pi\hbar)^d}f_{eq}\delta_{ij} = \rho\,\delta_{ij}$$

$$\langle\tau_{ij}|\varepsilon\rangle = \langle J^E_j|P_i\rangle = (\varepsilon+P)\delta_{ij}$$

Using thermodynamic identity $\varepsilon + P = \mu\rho + Ts$,

$$\tau_{ij} = \delta_{ij}\left[\rho\,\delta\mu + s\,\delta T\right] = \delta_{ij}\times\underbrace{\delta P}_{\text{thermodynamic pressure.}}$$

This confirms the results advertised in our earlier discussion on hydro. Now it is time to move to the derivative corrections to hydrodynamics... let’s write out the following

$$|\Phi(x,t)\rangle = \begin{pmatrix}|\Phi_s\rangle \\ |\Phi_f\rangle\end{pmatrix}\begin{array}{l}\leftarrow \text{“slow” hydro modes } |\rho^A\rangle \\ \leftarrow \text{“fast”, non-hydro modes } \langle\Phi_f|\rho^A\rangle = 0\end{array}$$

$$\partial_t \begin{pmatrix} |\Phi_s\rangle \\ |\Phi_f\rangle \end{pmatrix} + \begin{pmatrix} \vec{v}_{ss} & \vec{v}_{sf} \\ \vec{v}_{fs} & \vec{v}_{ff} \end{pmatrix} \cdot \nabla_x \begin{pmatrix} |\Phi_s\rangle \\ |\Phi_f\rangle \end{pmatrix} = - \begin{pmatrix} 0 & 0 \\ 0 & W_f \end{pmatrix} \begin{pmatrix} |\Phi_s\rangle \\ |\Phi_f\rangle \end{pmatrix}$$

If $\vec{v}_{sf} = 0$, then our ideal hydrodynamics is exact. But in general this is not true. Suppose $\| W_f |\Phi_f\rangle \| \geq \| \frac{1}{\tau_{int}} |\Phi_f\rangle \|$. If $\tau_{int}\partial_t \ll 1 \ldots$

$$\cancelto{0}{\partial_t |\Phi_f\rangle} + W_f |\Phi_f\rangle + \vec{v}_{fs} \cdot \nabla_{\vec{x}} |\Phi_s\rangle + \vec{v}_{ff} \cdot \nabla_{\vec{x}} |\Phi_f\rangle = 0.$$

on long length scales: $|\Phi_f\rangle \approx \underbrace{-W_f^{-1} \vec{v}_{fs} \cdot \nabla_{\vec{x}} |\Phi_s\rangle}_{\text{this term is suppressed by a } \underline{\text{derivative}}}$

$$\partial_t |\Phi_s\rangle + \vec{v}_{ss} \cdot \nabla_x |\Phi_s\rangle - \underbrace{(\vec{v}_{sf} \cdot \nabla_{\vec{x}}) W_f^{-1} (\vec{v}_{fs} \cdot \nabla_{\vec{x}}) |\Phi_s\rangle}_{\text{these are the dissipative coefficients, which scale as } \tau_{int} \text{ as promised!}} \approx 0$$

The entropy production rate is given by

$$T\dot{s} = \langle \Phi_f | W_f | \Phi_f \rangle = \langle \Phi_s | (\vec{v}_{fs}^{\,T} \cdot \overleftarrow{\nabla}_x) W_f^{-1} (\vec{v}_{fs} \cdot \nabla_{\vec{x}}) | \Phi_s \rangle$$

These derivative corrections serve to relax the system back to global equilibrium. In general these dissipative terms will be present.

3.4) Hydrodynamics of a low temperature Fermi liquid

Now let's run our algorithm on an isotropic Fermi liquid, assuming there are no impurities and all electron electron scattering is momentum conserving.

Using a relaxation time approximation, and assuming two dimensions for convenience...

$$|n,m\rangle = \int d^2p\, (|p|-p_F)^n e^{im\theta} |\vec{p}\rangle \quad \ldots \text{ (combination of previous bases...)}$$

$$\left.\begin{aligned} &\text{momentum: } |P_x\rangle \pm i|P_y\rangle = p_F|0,\pm1\rangle + |1,\pm1\rangle \\ &\text{number: } |n\rangle = |0,0\rangle \\ &\text{energy: } |\varepsilon\rangle = \mu|0,0\rangle + v_F|1,0\rangle + \cdots \end{aligned}\right\} \text{conserved quantities}$$

assume that:

$$W|\Phi\rangle = \frac{1}{\tau}|\Phi\rangle - \frac{1}{\tau}\sum_{\substack{\Psi=P_x,P_y,\\ \varepsilon,n}} \frac{\langle\Psi|\Phi\rangle}{\langle\Psi|\Psi\rangle}|\Psi\rangle, \text{ where } \tau \sim \frac{1}{T^2}$$

We follow the same procedure as before, keeping only terms relevant at the longest wavelengths...

Since $\langle n,m|n,m\rangle \sim T^{2n}$, approximately $n \geq 2$ modes unimportant.

Currents:

$$|J_x\rangle \pm i|J_y\rangle = v_F|0,\pm1\rangle + v_F'|1,\pm1\rangle + \cdots$$

$$|Q_x\rangle \pm i|Q_y\rangle \approx v_F^2|1,\pm1\rangle + \frac{\pi^2T^2}{3p_F}|0,\pm1\rangle + \cdots \qquad \mu|J_i\rangle + |Q_i\rangle = |J_i^E\rangle$$

$$|\tau_{xx}\rangle + |\tau_{yy}\rangle \approx \underbrace{a_1|0,0\rangle + a_2|1,0\rangle}_{\text{previously evaluated: } \approx \text{thermodynamic pressure!}} + \overset{O(T^2)}{a_3|2,0\rangle} + \cdots$$

$$(|\tau_{xx}\rangle - |\tau_{yy}\rangle) \pm i(|\tau_{xy}\rangle + |\tau_{yx}\rangle) \approx \frac{1}{2}p_Fv_F|0,\pm2\rangle$$

shear viscosity:

$$\eta = \left(\langle \tau_{xx}| - \langle \tau_{yy}|\right) W^{-1} \left(|\tau_{xx}\rangle - |\tau_{yy}\rangle\right)$$

$$= \left(\tfrac{1}{2} p_F v_F\right)^2 \tau \langle 0,2|0,2\rangle = \frac{v_F^2 \tau}{4} \times \frac{\nu}{2} p_F^2 \sim \frac{1}{T^2}$$

bulk viscosity:

$$\zeta \sim \tau \times \| \text{non-conserved part of } |\tau_{xx}\rangle + |\tau_{yy}\rangle \|$$

$$\sim \tau \times \| a_3 |2,0\rangle \|^2 \sim \tau \times \left(\frac{T}{T_F}\right)^4 \sim T^2$$

in a generic Fermi liquid $\boxed{\zeta \sim \eta \left(\frac{T}{T_F}\right)^4}$

incoherent conductivities? since $\left\| \frac{\langle P_x|Q_x\rangle}{\langle P_x|P_x\rangle} |P_x\rangle \right\| \sim T^2$ and $\| |Q_x\rangle \| \sim T$ the heat current is mostly incoherent, but the charge current $\| |J_x\rangle \| \sim T^0$, $\left\| |J_x\rangle - \frac{\langle P_x|J_x\rangle}{\langle P_x|P_x\rangle} |J_x\rangle \right\| \sim T$ is mostly coherent. We therefore expect that

$$\Sigma = \begin{pmatrix} \sigma_{\text{inc}} & T\alpha_{\text{inc}} \\ T\alpha_{\text{inc}} & T\bar{\kappa}_{\text{inc}} \end{pmatrix} \sim T^2 \tau$$

(only $\bar{\kappa}_{\text{inc}}$'s value is as large as might have been expected...)

$$\bar{\kappa}_{\text{inc}} \simeq \frac{\tau}{T} \langle Q_x | Q_x \rangle = \frac{\pi^2 T v_F^2 \nu \tau}{6} \sim \frac{1}{T}$$

To summarize what we learned, the linearized equations of hydrodynamics for a Fermi liquid are, using previously found identities

$$\langle J^E_i | P_i\rangle = \varepsilon + P, \quad \langle J_i | P_i\rangle = \rho, \quad \langle \tau_{ij}|\varepsilon\rangle = \delta_{ij}(\varepsilon+P), \quad \langle \tau_{ij}|n\rangle = \rho\delta_{ij};$$

$$|\Phi\rangle = \delta\mu|n\rangle + \delta u_i|P_i\rangle + \frac{\delta T}{T}\left(|\varepsilon\rangle - \mu|n\rangle\right)$$

$$\partial_t\delta\rho + \rho\,\partial_i\delta u_i \approx 0$$ [no incoherent charge current!]

$$\partial_t\underbrace{\left[\delta\varepsilon - \mu\delta\rho\right]}_{T\delta s} + \underbrace{\left[\varepsilon + P - \mu\rho\right]}_{Ts}\partial_i\delta u_i \approx \left.\bar{\kappa}_{inc}\,\partial_i\partial_i\delta T\right\}$$ these terms small/ignorable at low T!

bulk viscosity negligible

$$\partial_t\delta u_i\langle P_x|P_x\rangle + \partial_i\delta P = \eta\,\partial_j\left[\partial_j\delta u_i + \partial_i\delta u_j - \delta_{ij}\partial_k\delta u_k\right]$$

$$\langle P_x|P_x\rangle = \frac{\nu}{2}p_F^2 = n\frac{p_F}{v_F} = n\,m \leftarrow$$ effective mass!

Remarkably, these are (at leading order at low temperature...) precisely the hydrodynamic equations for a Galilean invariant fluid, where

$$\vec{J} = n\vec{u}$$ holds as an exact identity!

$$= \frac{1}{m}\vec{P} \leftarrow$$ momentum density

The Galilean symmetry forbids all the incoherent conductivities except the thermal one in our standard fluid frame, because the current and momentum density are exactly related!

Remarkably we did NOT assume Galilean invariance in our theory, but for any isotropic Fermi liquid there is a sense in which this symmetry "approximately" arises at low T. One way to think about this is that at low T, we can't tell what the full dispersion relation is. If it was Galilean ...

$$\varepsilon(p) = \frac{p^2}{2m}$$

Then all of the identities we wrote would have to hold exactly. The breaking of Galilean symmetry is at the same order where we can see deviations from the quadratic dispersion...which means we have to go to O(T^4) in dissipative coefficients !

3.5) Quasinormal modes

Reference: 1704.07384

The simplest question we can ask about these fluids is to study plane wave solutions to the linearized EOMs:

$$\text{e.g.}\quad \begin{pmatrix} \delta\mu \\ \delta T \\ \delta u_i \end{pmatrix} \sim e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

The simplest fluid is the one with only scalar conserved quantities, where we found

$$\partial_t \rho_A = D_{AB} \nabla^2 \rho_B \implies \underbrace{-i\omega \rho_A = -k^2 D_{AB} \rho_B}\ ?$$

$$\omega = -i D_\alpha k^2$$

D_α ← eigenvalue of D_{AB}, corresponding to eigenvector ρ^α_A

Stability $\implies$ $\mathrm{Re}(D_\alpha) \geq 0$ so that $\mathrm{Im}(\omega) \leq 0$

Since D is a <u>real</u> matrix, this is equivalent to D being positive semidefinite

Next, let's consider a fluid with conserved momentum. In this case,

$$-i\omega \underbrace{(\mathcal{M}\,\delta v_i)}_{\substack{\text{momentum density,} \\ \mathcal{M} \text{ is some pheno prefactor}}} + ik_i \underbrace{\delta P}_{\delta P = \rho_A \delta\mu_A} + \eta k_j (k_i \delta v_j + k_j \delta v_i) + \left(\zeta - \frac{2}{d}\eta\right) k_i k_j \delta v_j = 0$$

$$-i\omega \chi_{AB}\, \delta\mu_B + ik_i \rho_A\, \delta v_i + k^2 \Sigma_{AB}\, \delta\mu_B = 0$$

$\chi_{AB} = \dfrac{\partial \rho_A}{\partial \mu_B}$

ρ_A ← same coefficient!

1) If $\vec{k} \perp \delta\vec{v}$: $\quad \delta\mu_A = 0$ and

$$-i\omega \mathcal{M}\, \delta v_\perp + \eta k^2 \delta v_\perp = 0, \qquad \underbrace{\omega = -i \frac{\eta}{\mathcal{M}} k^2}_{\text{diffusion of transverse momentum!}}$$

2) If $\vec{k} \parallel \delta\vec{v}$ and $\delta\vec{v} \neq 0$

$$-i\omega\mathcal{M}\delta v + ik\rho_A \delta\mu_A + k^2\left(2\frac{d-1}{d}\eta + \zeta\right)\delta v_i = 0$$

$$\left(k^2\Sigma_{AB} - i\omega\chi_{AB}\right)\delta\mu_B = -ik\rho_A \frac{-ik\rho_B\delta\mu_B}{-i\omega\mathcal{M} + k^2\left(2\frac{d-1}{d}\eta + \zeta\right)}$$

Since $\delta v \neq 0$, $\rho_B\delta\mu_B \neq 0$. Therefore,

$$1 = \frac{-k^2}{-i\omega\mathcal{M} + k^2\left(2\frac{d-1}{d}\eta + \zeta\right)} \rho_A \left(k^2\Sigma_{AB} - i\omega\chi_{AB}\right)^{-1}\rho_B$$

Claim. as $k \to 0$, $\omega \sim \pm v_s k$: [sound waves!]

$$1 \approx \frac{k^2}{i\omega\mathcal{M}} \times \frac{1}{-i\omega} \rho_A \chi^{-1}_{AB}\rho_B, \qquad v_s = \sqrt{\frac{\rho_A\chi^{-1}_{AB}\rho_B}{\mathcal{M}}}$$

More carefully:

$$\omega = \pm v_s k - i\Gamma k^2 + O(k^3), \text{ where } \Gamma = \frac{2\frac{d-1}{d}\eta + \zeta}{\mathcal{M}} + \frac{\rho_A\chi^{-1}_{AC}\Sigma_{CD}\chi^{-1}_{DB}\rho_B}{\mathcal{M}v_s^2}$$

3) If $\delta\vec{v} = 0$ but $\delta\mu_A \neq 0$: $\quad \rho_A\delta\mu_A = 0$

Define a matrix $P_{AB} = \delta_{AB} - \dfrac{\rho_A\rho_C\Sigma^{-1}_{CB}}{\rho\cdot\Sigma^{-1}\rho}$

$$P_{CA}\omega\chi_{AB}\delta\mu_B = -ik^2 P_{CA}\Sigma_{AB}\delta\mu_B = -ik^2\left(\Sigma_{AB}\delta\mu_B - \frac{\rho_A\rho_B\delta\mu_B}{\rho\cdot\Sigma^{-1}\rho}\right)$$

$$= -ik^2\Sigma_{CA}\left(P^T\right)_{AB}\delta\mu_B$$

Suppressing matrix indices...

$$\omega\times\Sigma^{-1}P\chi\delta\mu = -ik^2P^T\delta\mu$$

$$= \omega\times\underbrace{P^T\Sigma^{-1}P\chi}_{=D^{-1}}\delta\mu$$

D^{-1} has a null vector: $\rho_A D^{-1}_{AB} = 0$.

The remaining eigenvalues correspond to diffusion modes $\omega = -iD_\alpha k^2$

In a generic fluid, all of the modes will be either sound waves or diffusion modes (at least in the presence of time reversal symmetry and inversion symmetry). Reference: 1710.11141

3.6) Plasmons

These sound and diffusion modes are commonly observed in all kinds of classical fluids. But why not in electron fluids? There are two obvious issues. Firstly, the presence of impurities leads to momentum relaxation, as we will discuss in a little bit. But for propagating sound modes, there is also the problem of the long range Coulomb interactions in metals...

For simplicity let's focus on the sector of charge and momentum dynamics, relevant for low T...

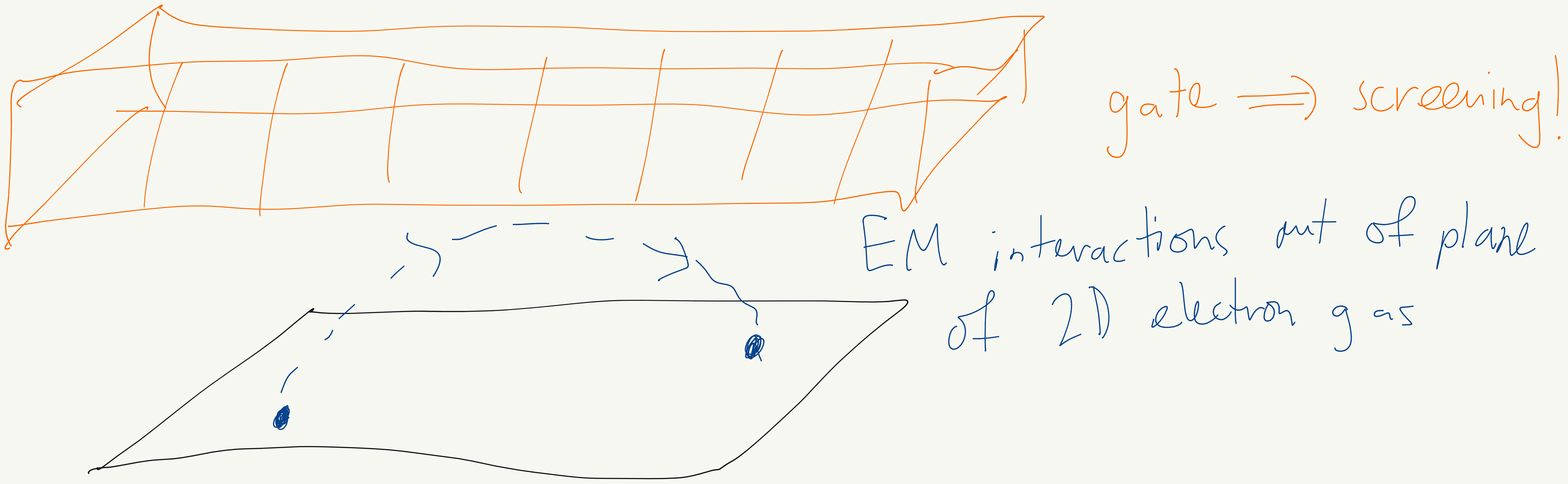
$$\partial_t n + n\partial_j u_j = 0$$

Coulomb!

$$mn\partial_t u_i + n\partial_i\mu - \eta\partial_j\partial_j u_i = \rho F_{ext}$$

$$F_{ext} = -\partial_i\int d^dy\,\frac{e^2}{|x-y|}(\rho(y)-\rho_0),$$ from Gauss' Law!

Whether the electrons are confined to 2d or not, we have the 1/r potential...except in the presence of a gate!



Let's again look for quasinormal modes. We need to Fourier transform the Coulomb interaction

$$\int d^d x \, e^{-ik\cdot\vec{x}} \int d^d y \, \frac{e^2}{|x-\vec{y}|} \int \frac{d^d\vec{q}}{(2\pi)^d} e^{i\vec{q}\cdot\vec{y}} \delta\rho(\vec{q})$$

$$= \delta\rho(\vec{k}) \underbrace{\int d^d\vec{y} \, \frac{e^2}{|\vec{y}|} e^{i\vec{k}\cdot\vec{y}}}$$

In $d=3$:

$$2\pi \int_0^\pi d\theta \int_0^\infty dy \sin\theta \, \frac{e^{iky\cos\theta}}{y} e^{-\lambda y} y^2 = 2\pi \int_0^y dy \, \frac{1}{ik}\left[e^{iky} - e^{-iky}\right] e^{-\lambda y}$$

($\lambda \to 0$ is a regulator)

$$= \frac{2\pi}{ik}\left[\frac{1}{\lambda - ik} - \frac{1}{\lambda + ik}\right] = \frac{2\pi}{ik}\frac{2ik}{\lambda^2 + k^2} = \frac{4\pi}{k^2} \quad (\lambda \to 0)$$

In $d=2$:

$$\int_0^{2\pi} d\theta \int_0^\infty dy \, y \, \frac{e^{iky\cos\theta}}{y} = \int_0^\infty dy \int_0^{2\pi} d\theta \, e^{iky\cos\theta} = 2\pi \int_0^\infty dy \, J_0(ky)$$

$$= \frac{2\pi}{|\vec{k}|}$$

Now we plug back in to the equation of motion

$$\underbrace{-i\omega\nu\delta\mu}_{\partial_t\delta\rho} + ik_i n\delta u_i = 0$$

$\frac{4\pi e^2}{k^2}$ in $d=3$, or $\frac{2\pi e^2}{|k|}$ in $d=2$

$\downarrow$

$$-i\omega mn\,\delta u_i + ik_i n\delta\mu + \eta k^2\delta u_i + ik_i U(k)\nu n\delta\mu = 0$$

$$\Rightarrow 0 = -\omega mn - i\eta k^2 + \frac{n^2}{\nu}\frac{k^2}{\omega}\left(1+\nu U(k)\right)$$

let $\tilde{v}_s^2 = \frac{n}{m\nu} + \frac{n}{m}U(k)$ be effective sound speed...

$$\omega = \pm\tilde{v}_s k - i\frac{\eta}{2mn}k^2 + \dots$$

Since $\nu = d\times\frac{n}{p_F v_F}\dots$

In $d=3$: $\omega = \pm\sqrt{\frac{v_F^2}{3}k^2\left(1+\frac{4\pi e^2}{k^2}\nu\right)} + \dots$

$$\approx \pm\sqrt{\frac{4\pi e^2}{m}n} = \pm\omega_p \text{ (plasma frequency)}$$

In a typical metal, $\omega_p = 10^{15}\,s^{-1} = \frac{1}{1\text{ fs}}$

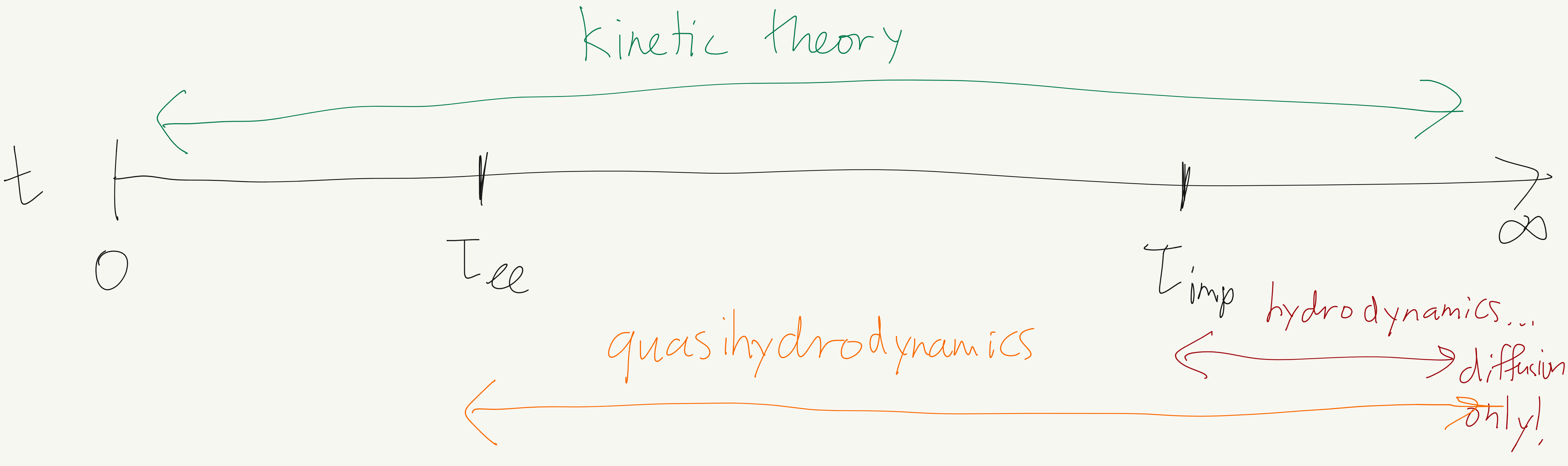
This scale is usually very fast compared to electron-electron scattering, so well beyond the “hydrodynamic” frequency scales. Thus there are no propagating sound modes.

$d=2$: $\omega = \pm\sqrt{\frac{2\pi e^2 n}{m}|k|} - i\frac{\eta}{2mn}k^2 + \dots$

Plasmons propagate much faster than ordinary sound modes, but still with a finite velocity. In either dimension, the plasmon dispersion is robust against hydrodynamic effects up to its decay rate. The real part of these dispersion relations has been seen in many metals, but the imaginary part is harder to detect, and there are of course other contributions to plasmon decay...

3.7) Momentum relaxation and quasihydrodynamics

The second issue in a metal that we need to deal with is the presence of impurities which relax momentum. If the impurity scattering rate is extremely long lived though, we can incorporate it...



Quasihydrodynamics is a label that we assign to a theory where some of the degrees of freedom are not exactly conserved, but whose decay rates are extremely small. It is easiest to think about with an example — in our case, the Fermi liquid hydrodynamics in the presence of momentum relaxation...

$$\partial_t n + n\partial_i u_i = 0$$

$$mn\partial_t u_i + n\partial_i \mu - \eta\partial_j\partial_j u_i \approx -\frac{mn}{\tau_{imp}} u_i$$

leads to quasihydro!

Let's first think about the quasinormal modes here. We find that

$$-i\omega\nu\delta\mu + n ik\delta u = 0$$

$$-i\omega\delta u + \frac{ik}{m}\delta\mu + \frac{\eta}{mn}k^2\delta u + \frac{1}{\tau_{imp}}\delta u = 0$$

$$\Rightarrow 0 = \omega + i\left(\frac{1}{\tau_{imp}} + \frac{\eta k^2}{mn}\right) - \frac{k^2}{\omega}v_s^2, \quad \text{where} \quad v_s^2 = \frac{n}{\nu m} \sim v_F^2$$

$$\omega = \frac{1}{2}\left[-\frac{i}{\tau_{imp}} - \frac{i\eta k^2}{mn} \pm \underbrace{\sqrt{4v_s^2k^2 - \left(\frac{1}{\tau_{imp}} + \frac{\eta k^2}{mn}\right)^2}}_{\approx\sqrt{4v_s^2k^2 - \frac{1}{\tau_{imp}^2}}}\right]$$

for $\frac{1}{\tau_{imp}}$ small...

When $v_s k \gg \frac{1}{\tau_{imp}}$, sound waves...

$$\omega \approx \pm v_s k - i\left(\frac{1}{2\tau_{imp}} + \frac{\eta k^2}{2mn}\right) + \dots$$

usual decay of sound

extra decay coming from momentum relax.

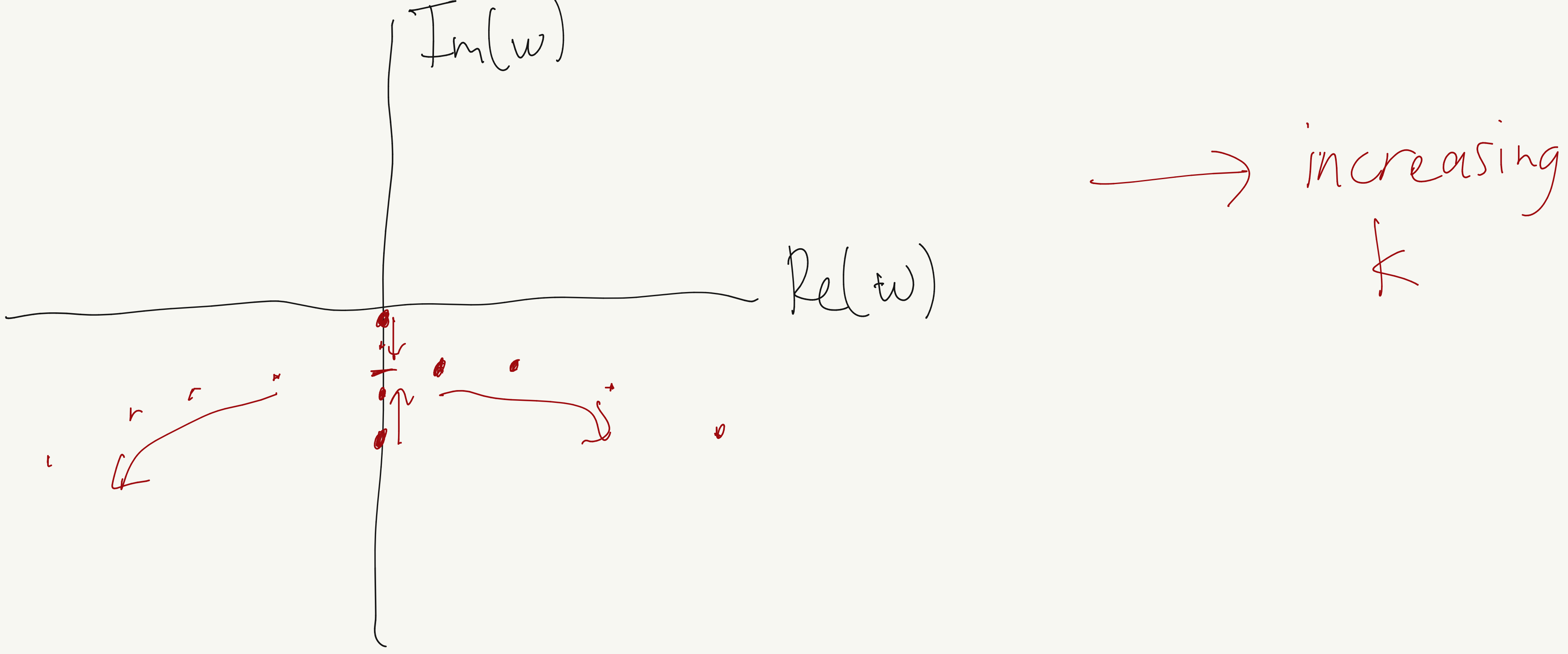
But when $v_s k \ll \frac{1}{\tau_{imp}}$ ...

gapped momentum mode

$$\omega \to \begin{cases} -i/\tau_{imp} + \dots \\ -\frac{i}{2\tau_{imp}}\left(1 - \sqrt{1-(2v_s\tau_{imp}k)^2}\right) \approx -i v_s^2 \tau_{imp} k^2 \end{cases}$$

Ohmic charge diffusion!

It is instructive to draw this by going into the complex plane:



The “quasihydrodynamic” momentum decay mode “collides” with the charge diffusion mode to create two sound modes. For a second perspective on this, let’s think about the equations of motion. For simplicity let’s imagine static flows

$$\cancelto{0}{\partial_t u_i} + \frac{1}{m}\partial_i\mu - \frac{\eta}{mn}\partial_j\partial_j u_i + \frac{u_i}{\tau_{imp}} = 0 \qquad \rho\,\partial_i u_i = 0$$

For a circular Fermi surface, using identities derived previously...

$$\frac{1}{m}\partial_i\mu - \frac{v_F^2\tau_{ee}}{4}\partial_j\partial_j u_i + \frac{u_i}{\tau_{imp}} = 0$$

Define Gurzhi length $\lambda = \frac{1}{2}v_F\sqrt{\tau_{ee}\tau_{imp}}$

On scales $\Delta x \ll \lambda$:

$$\frac{1}{m}\partial_i \mu \approx \frac{\eta}{mn}\partial_j \partial_j u_i \qquad \text{[momentum relaxation not important]}$$

but on scales $\Delta x \gg \lambda$:

$$\frac{1}{m}\partial_i \mu \approx -\frac{u_i}{\tau_{imp}}$$

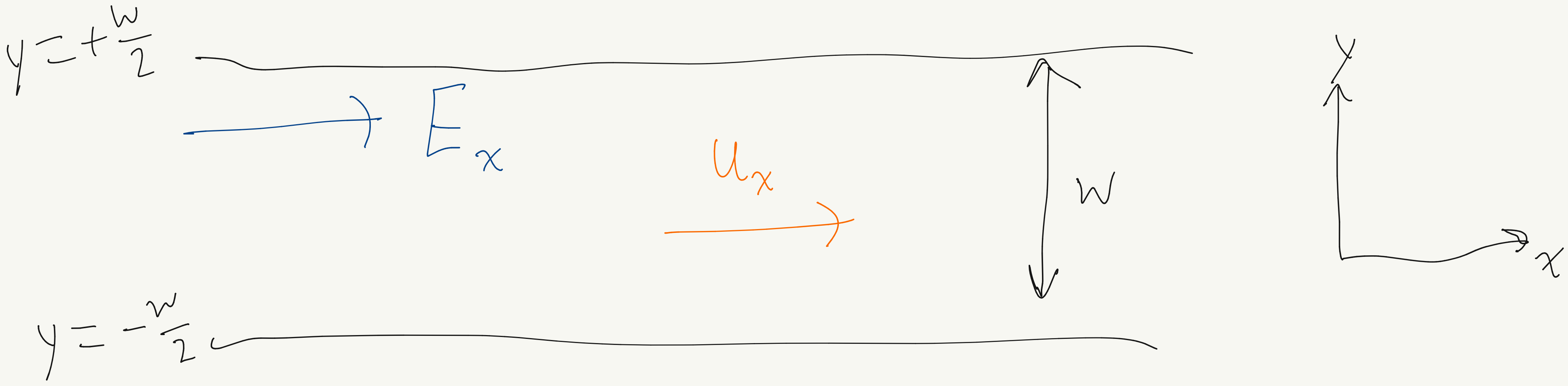
$$\partial_i(n u_i) = 0 = \underbrace{\partial_i\left(-\frac{n\tau_{imp}}{m}\partial_i \mu\right)} = 0$$

$$\partial_i\left(-\frac{ne^2\tau_{imp}}{m}\partial_i \mu\right) = \underbrace{\partial_i\left(-\sigma\,\partial_i \mu\right)}_{\text{Ohmic transport!}} = 0$$

Roughly speaking, viscous electron flows become relevant on scales short compared to lambda, but still long compared to the electron scattering length...

3.8) Transport in narrow channels

Let’s give a specific example of this effect.   Consider the following setup



The relevant hydrodynamic equations here are...(assuming solutions only depend on the y coordinate)

$$\partial_y(n u_y) = 0 \qquad -enE_x = -\eta\,\partial_y^2 u_x + \frac{mn}{\tau_{imp}} u_x$$

Let’s assume boundary conditions

$$\underbrace{u_y = 0}_{\text{no current flows through walls!}} \text{ at } y = \pm\frac{w}{2}, \text{ and } \qquad u_x = \underset{\text{slip length.}}{\zeta}\,\frac{\partial u_x}{\partial n} \leftarrow \text{normal derivative}$$

The precise slip length is relatively unclear in current experiments, so we will treat it as a free parameter.

We find that $u_y = 0$ and

using symmetry $y \to -y$

$$\frac{enE_x}{\eta} = \left(\partial_y^2 - \frac{1}{\lambda^2}\right) u_x \implies u_x = \underbrace{-\frac{\lambda^2 enE_x}{\eta}}_{u_0} + A\cosh\frac{y}{\lambda}$$

$$\text{At } y = -\frac{w}{2}: \quad A\cosh\frac{(-w)}{2\lambda} + u_0 = \ell \times \frac{A}{\lambda}\sinh\frac{(-w)}{2\lambda}$$

$$\implies -u_0 = A\left(\cosh\frac{w}{2\lambda} + \frac{\ell}{\lambda}\sinh\frac{w}{2\lambda}\right)$$

$$u_x = u_0\left[1 - \frac{\cosh y/\lambda}{\cosh\frac{w}{2\lambda} + \frac{\ell}{\lambda}\sinh\frac{w}{2\lambda}}\right]$$

Let’s finally return to a transport calculation, and compute the resistance per unit length of the channel...!

$$I = -en\int_{-w/2}^{w/2} dy\, u_x(y) = \frac{ne^2\tau_{imp}}{m} E_x \int_{-w/2}^{w/2} dy \left[1 - \frac{\cosh y/\lambda}{\cosh\frac{w}{2\lambda} + \frac{\ell}{\lambda}\sinh\frac{w}{2\lambda}}\right]$$

impurity limited conductivity!

$$= \sigma_{imp} E_x \times \left[w - \frac{2\lambda\sinh\frac{w}{2\lambda}}{\cosh\frac{w}{2\lambda} + \frac{\ell}{\lambda}\sinh\frac{w}{2\lambda}}\right] \leftarrow \text{geometric factor!}$$

$$\text{If } w \gg \lambda: \quad I \approx \sigma_{imp} E_x \times \left[w - \frac{2\lambda^2}{\lambda + \ell}\right] \approx \sigma_{imp} E_x w$$

This is the same transport coefficient as if we just solved the Ohmic transport equations directly!

Ohmic resistance

per length

$$\frac{R}{L} = \frac{V}{L} \times \frac{1}{I} = \frac{E_x}{I} = \frac{1}{w\sigma_{imp}}$$

In contrast, if the Gurzhi length is very large compared to the channel width...

$$I \simeq \sigma_{imp} E_x \times w \left[1 - \frac{1 + \frac{1}{6}\left(\frac{w}{2\lambda}\right)^2 + \cdots}{1 + \frac{1}{2}\left(\frac{w}{2\lambda}\right)^2 + \frac{\zeta w}{2\lambda^2}} \right] \cdots$$

if ζ relatively small...

$$\simeq \sigma_{imp} E_x w \left[\frac{\zeta w}{2\lambda^2} + \frac{w^2}{12\lambda^2} + \cdots \right]$$

$$= \frac{e^2 n^2}{12\eta} E_x w^2 (w + 6\zeta) \Longrightarrow \frac{R}{L} = \frac{12\eta}{e^2 n^2} \frac{1}{w^2(w+6\zeta)}$$

In particular, when $w \gg \zeta$ ... $\frac{R}{L} = \frac{12\eta}{e^2 n^2 w^3}$

This very peculiar scaling of the channel resistivity is, in principle, a clear prediction of the hydrodynamic flow regime. In practice, as we will discuss more soon, it is not so easy to see in experiments...

Also note that because this prediction of hydrodynamics is not particularly sensitive to electronic dynamics, only to the momentum conservation equation, one could also predict such behavior for phonons in the presence of an applied temperature gradient, where the w^3 scaling would again be a signature of hydro

One last very interesting point here. We’ve found that in the regime where channel is small enough that electrons behave like a viscous fluid,

$$\frac{R}{L} \sim \eta \sim \tau_{ee}$$

The faster the collision rate, the smaller the resistance! This violates one of our “theorems” in the kinetic theory of transport, but in reality it is simply pointing to the breakdown of some of the assumptions that we made there. In particular, spatial inhomogeneity of the channel was crucial to get this effect to work.

Finally, if the slip length $\zeta \to \infty$ ....

$I = \sigma_{imp} E_x w$ independent of w/λ.

What is going on here? If the slip length is infinite, the boundary condition is that

$\partial_y u_x = 0 \Longrightarrow \tau_{xy} = 0 \Rightarrow$ no momentum flux out of electron fluid.

It was crucial, to get this modified viscous resistance, that the boundary conditions on the fluid relaxed momentum. If they conserve momentum, then resistivity is always given by the Ohmic answer, even for a short channel!

3.9) Transport in inhomogeneous media

Reference: 1011.3068

More generally, let us calculate the transport properties of an electron fluid moving in a highly inhomogeneous medium. We will assume that the fluid is almost homogeneous, except for the presence of weak potential inhomogeneity...

The fully nonlinear hydrodynamic equations are, assuming no time dependence...

$$\partial_i\left[\rho^A u_i + \Sigma^{AB}\left(E_i^B - \partial_i\mu^B\right)\right] = 0,$$

where E_i^B are external sources $\begin{pmatrix} E \\ -\nabla T \end{pmatrix}$ for $\underset{\uparrow\ \text{internal fluid variables...}}{\begin{pmatrix} \mu \\ T \end{pmatrix}}$

$$\rho^A\left(\partial_i\mu^A - E_i^A\right) - \partial_j\left[\eta\left(\partial_i u_j + \partial_j u_i\right)\right] - \partial_i\left[\left(\zeta - \frac{2\eta}{d}\right)\partial_j u_j\right] = 0$$

Up to higher derivative corrections. Observe that the sources and thermodynamic variables always come coupled together. This is important, and it represents that the fluid cannot locally distinguish between whether a thermodynamic gradient comes from the fluid itself or an external source. Let us imagine applying the following external source:

$$E_i^A = \underbrace{\delta E_i^A}_{\substack{\text{infinitesimal electric field} \\ \text{e.g. for transport}}} + \underbrace{\partial_i \mu^A_{dis}(x)}_{\substack{\text{Spatially inhomogeneous, small} \\ \text{amplitude but NOT infinitesimal.}}}$$

In equilibrium (i.e. $\delta E_i^A = 0$); $\mu^A(x) = \mu^A_{dis}(x)$

To solve at linear order in δE_i^A...

Let $\mu^A_{dis} \sim \lambda$ (small parameter). Take $\delta E \to 0$ THEN $\lambda \to 0$...

Claim. If $u_i = \overset{\mathcal{O}(\delta E)}{\delta u_i}$, $\mu^A = \delta\mu^A$, then at leading order:

$$\delta u_i = \frac{\overset{\text{constant}}{\delta\bar{u}_i}}{\lambda^2} + \frac{\delta\tilde{u}_i(x)}{\lambda} + \cdots$$

$$\delta\mu^A = \frac{\delta\tilde{\mu}^A(x)}{\lambda} + \cdots$$

Take the Fourier transform of the perturbations. We find that

← constant, background/equilibrium value

$$ik_i\left[\rho^A \delta u_i(k) - i k_i \Sigma^{AB}\delta\mu^B(k) + \overbrace{\frac{\partial\rho^A}{\partial\mu^B}\mu^B_{dis}}^{\rho^A_{dis}}\delta\bar{u}_i\right] = 0$$

$$\rho^A ik_i\delta\mu^A + \eta k^2 \delta u_i + \left(\zeta + \frac{d-2}{d}\eta\right)k_ik_j\delta u_j = 0$$

$\Big\}$ at $\mathcal{O}(\lambda^{-1})$

$$k_j\delta u_j\left[\zeta + \frac{2d-2}{d}\eta\right]\cancel{k^2} = -i\cancel{k^2}\rho^A\delta\mu^A$$

$$\frac{\rho^A\rho^B\delta\mu^B}{\zeta+\frac{2d-2}{d}\eta} + k^2\Sigma^{AB}\delta\mu^B + ik_i\delta\bar{u}_i\rho^A_{dis} = 0.$$

$$\delta\mu^A(k) = \left[\frac{\rho^A\rho^B}{\zeta+\frac{2d-2}{d}\eta} + k^2\Sigma^{AB}\right]^{-1}(ik_i\delta\bar{u}_i)\rho^B_{dis}$$

Integrate momentum equation at $\mathcal{O}(\lambda^0)$:

$$\rho^A E^A_i = \int\frac{d^dk}{(2\pi)^d}\rho^A_{dis}(k)\, ik_i\, \delta\mu^B(-k)$$

$$= \left(\int\frac{d^dk}{(2\pi)^d}\rho^A_{dis}(k)\, k_ik_j\, \rho^B_{dis}(-k)\left[\frac{\rho^A\rho^B}{\zeta+\frac{2d-2}{d}\eta} + k^2\Sigma^{AB}\right]^{-1}\right)\delta\bar{u}_j$$

$= \Gamma_{ij}\delta\bar{u}_j$, where Γ_{ij} ~ "momentum relaxation" tensor...

At leading order, $J^A_i = \rho^A\delta\bar{u}_i \Rightarrow \boxed{\sigma^{AB}_{ij} = \rho^A\Gamma^{-1}_{ij}\rho^B}$

We have just derived the precise form of the "hydrodynamic transport coefficients" found in our earlier treatment of transport at strong momentum conserving scattering rates. Namely, all transport coefficients are proportional to thermodynamic densities, weighted by an inverse momentum relaxation tensor.

Let's now focus our discussion on the theory of transport in a Fermi liquid at low temperatures, where we have

$$\Gamma_{ij} = \int \frac{d^dk}{(2\pi)^d} k_i k_j \begin{pmatrix} -en_{dis} \\ Ts_{dis} \end{pmatrix} \left[\begin{pmatrix} e^2n^2 & -enTs \\ -enTs & T^2s^2 \end{pmatrix} \frac{1}{\rho + \frac{2d-2}{d}\eta} + \begin{pmatrix} 0 & 0 \\ 0 & T\bar{\kappa}_0 k^2 \end{pmatrix} \right]^{-1} \begin{pmatrix} -en_{dis} & Ts_{dis} \end{pmatrix}$$

$$= \int \frac{d^dk}{(2\pi)^d} k_i k_j \begin{pmatrix} -en_{dis} \\ Ts_{dis} \end{pmatrix} \begin{pmatrix} \frac{\rho + \frac{2d-2}{d}\eta}{e^2n^2} + \left(\frac{Ts}{en}\right)^2 \frac{1}{T\bar{\kappa}_0 k^2} & \frac{Ts}{en}\frac{1}{T\bar{\kappa}_0 k^2} \\ \frac{Ts}{en}\frac{1}{T\bar{\kappa}_0 k^2} & \frac{1}{T\bar{\kappa}_0 k^2} \end{pmatrix} \begin{pmatrix} -en_{dis} & Ts_{dis} \end{pmatrix}$$

$$= \int \frac{d^dk}{(2\pi)^d} \left\{ k_i k_j \left| \frac{n_{dis}(k)}{n} \right|^2 \left(\rho + \frac{2d-2}{d}\eta\right) + \frac{T k_i k_j e^2}{\bar{\kappa}_0 k^2 n^2} \left| s n_{dis}(k) - n s_{dis}(k) \right|^2 \right\}$$

If the disorder varies on length scale $\xi \gg l_{ee}$: $\left(k \lesssim \frac{1}{\xi}\right)$:

electrical resistivity $\rho \sim \frac{1}{e^2n^2}\frac{\eta}{\xi^2} + \frac{T s^2}{\bar{\kappa}_0 n^2}$

$$\sim \frac{1}{T^2\xi^2} + \frac{T^4}{\xi^0}$$

analogous to the viscous effects we've seen in the channel

contribution to transport from heat diffusion ... would dominate deep in hydro limit