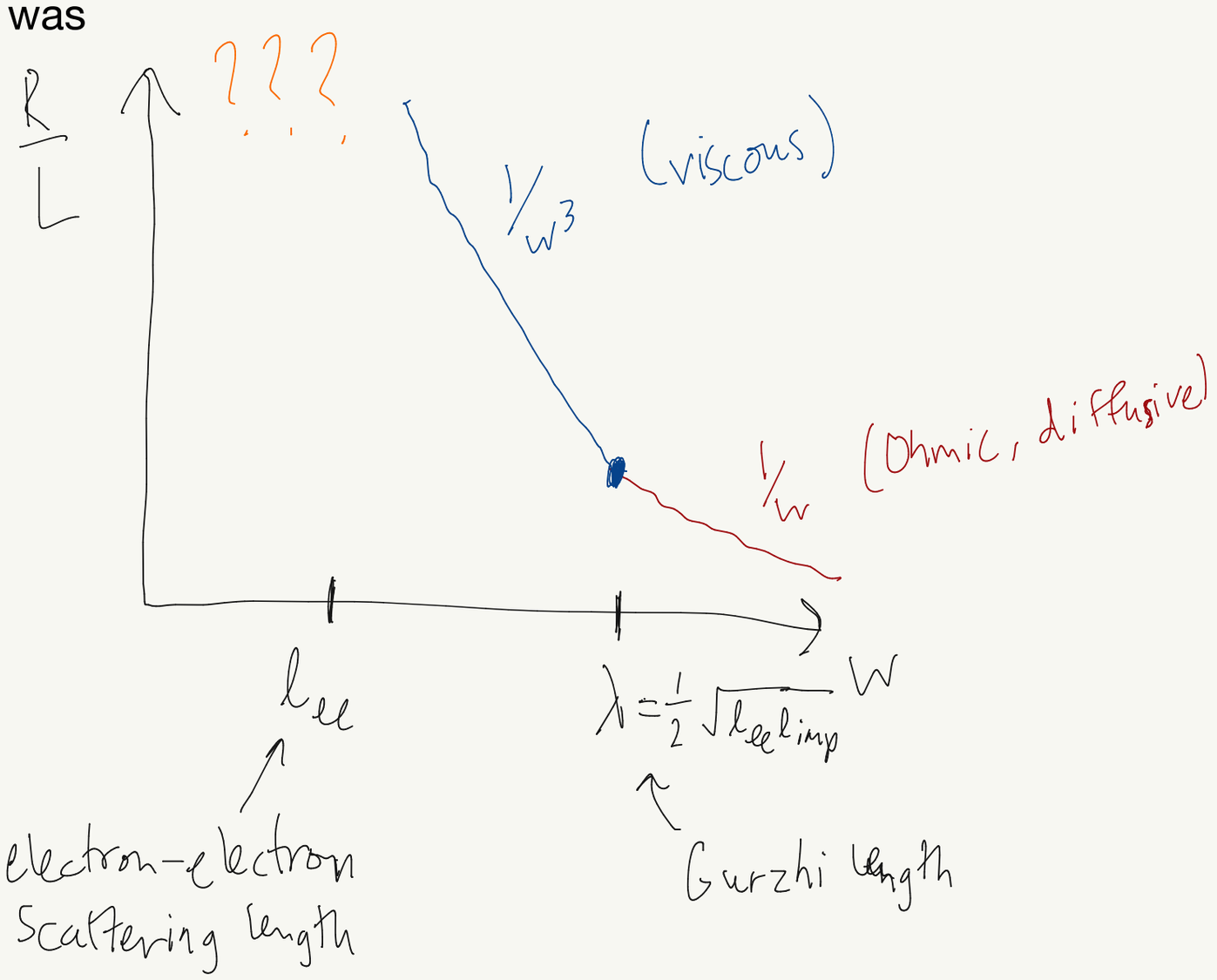
# 4. The ballistic-to-hydrodynamic crossover

## 4.1) Ballistic transport in narrow channels

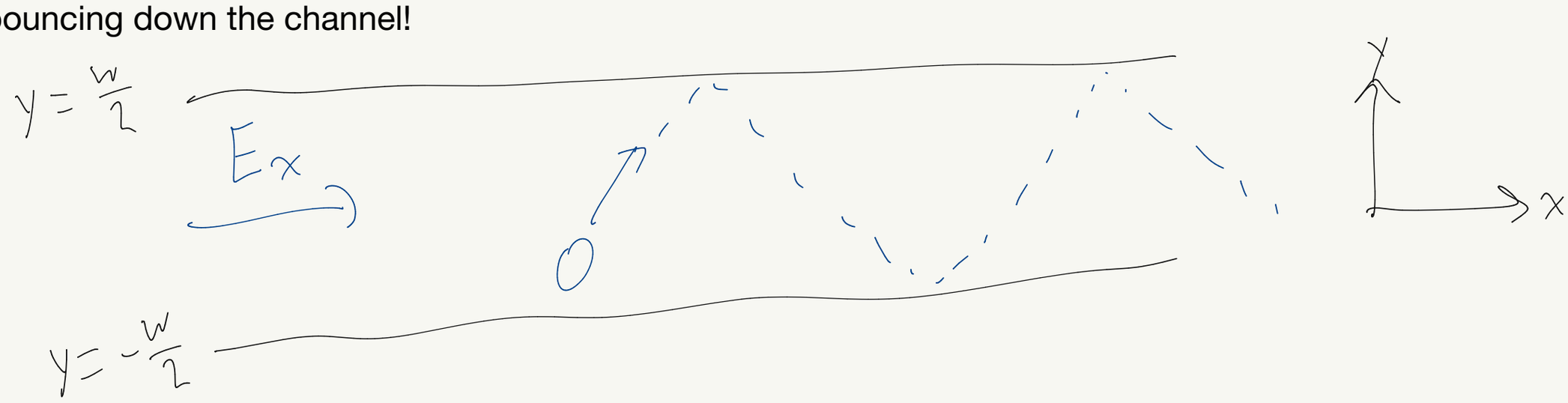
Reference: 1801.02879

Let's recall where we ended before. We found that the resistivity of an electron fluid in a narrow channel was




What happens when $w \ll l_{ee}$?

When the channel length is so short, we just expect there to be no collisions whatsoever! This is called the ballistic regime, and we can access it by solving the Boltzmann equation with no collisions — i.e., a collection of non-interacting particles bouncing down the channel!




Before doing a detailed calculation, let's estimate what we expect.

$$\frac{R}{L} \sim \frac{1}{w} \times \frac{1}{\tau}$$

"parallel resistors"

$\frac{1}{\tau}$: momentum relaxation rate per width ...

Ohmic: $\frac{1}{\tau} = \frac{1}{\tau_{imp}} = \text{constant!}$

$R \sim \frac{1}{w}$

viscous: $\frac{1}{\tau} \sim D_{mom}\left(\frac{1}{w}\right)^2 \sim \frac{\eta}{nmw^2}$

$R \sim \frac{1}{w^3}$

ballistic: $\frac{1}{\tau} \sim \frac{v_F}{w}$ $\quad R \sim \frac{1}{w^2}$

Put another way, we predict that the momentum relaxation rate is limited by the time it takes for the electron to bounce back and forth across the channel.

Now let's do a calculation. We will go ahead and use the Boltzmann equation...

$$\vec{v}\cdot\nabla_x f - e\vec{E}\cdot\partial_{\vec{p}} f = 0 \quad \dots \quad \text{if} \quad f = f_F(\varepsilon) - \frac{\partial f_F}{\partial \varepsilon}\Phi(\vec{p},\vec{x}) + \cdots$$

$$\left(-\frac{\partial f_F}{\partial \varepsilon}\right)\vec{v}\cdot\nabla_x\Phi = -e\vec{E}\cdot\vec{v}\left(-\frac{\partial f_F}{\partial \varepsilon}\right)$$

As before, assume $\Phi(\vec{x},\vec{p}) = \Phi(y, p_x, p_y)$ [homogeneity in the x-direction]

$$v_y(\vec{p})\,\partial_y\Phi = -eE_x v_x(\vec{p}).$$

[for simplicity, let's consider $d=2$]

At low temperatures in a Fermi liquid, we need only keep track of the fluctuations right at the Fermi surface. So we conclude that it's safe to write

$$\Phi(\vec{p},y) \simeq \Phi(|\vec{p}|=p_F,\theta,y)$$

"angle" on Fermi surface...

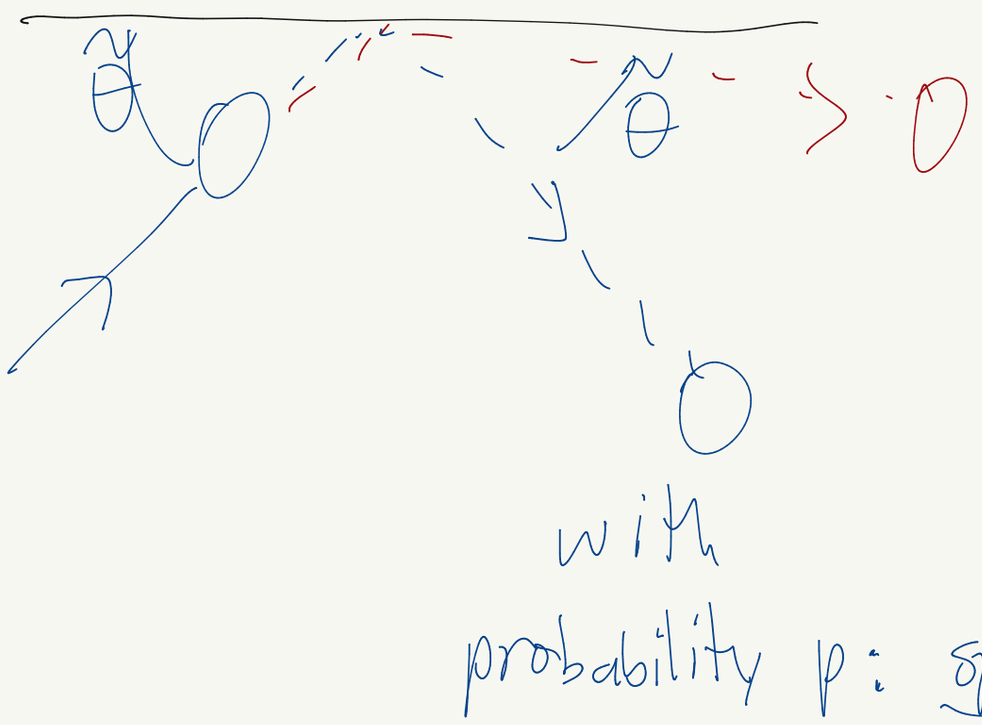
$$\partial_y\Phi = -eE_x\frac{v_x}{v_y}$$

For a circular Fermi surface...

$$\partial_y\Phi = -eE_x\cot\theta$$

$$\Phi(y,\theta) = \Phi(0,\theta) - (eE_x\cot\theta)\,y$$

Now comes the hard part — we need to impose boundary conditions! Let's do the following...




p is a measure of the roughness of the atomic boundary (?)

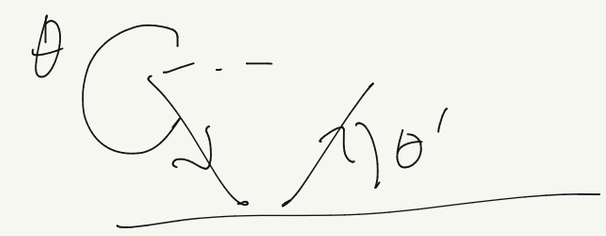
At the boundary $y = +\frac{w}{2}$ ...


incoming particles have angles $0 < \theta < \pi$

specular reflection at

$$\theta' = -\theta$$

at $y = -\frac{w}{2}$




incoming at $-\pi < \theta < 0$

specular reflection at

$$\theta' = -\theta$$

For the kinetic problem to be sensible physically, we'll impose boundary conditions relating the outgoing distribution to the incoming distribution.

At $y = +\frac{w}{2}$: $\Phi(-\theta) = p\,\Phi(\theta) + (1-p)\int_0^{\pi} \frac{d\phi}{\pi}\,\Phi(\phi)$ $\quad (\theta > 0)$

$y = -\frac{w}{2}$: $\Phi(-\theta) = p\,\Phi(\theta) + (1-p)\int_{-\pi}^{0} \frac{d\phi}{\pi}\,\Phi(\phi)$ $\quad (\theta < 0)$

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Define $g(\theta) = \Phi\left(\theta, -\text{sign}(\theta)\frac{w}{2}\right)$, so that

$$\Phi(\theta, y) = g(\theta) - eE_x \cot\theta \left(y - \frac{w}{2}\,\text{sign}(\theta)\right)$$

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$\theta > 0$: $g(-\theta) = p\left[g(\theta) + a\cot\theta\right] + (1-p)\int_0^{\pi} \frac{d\phi}{\pi}\left[g(\phi) + a\cot\phi\right]$ (0 by symmetry)

where $a = -eE_x w$

$$g(\theta) = p\left[g(-\theta) + a\cot\theta\right] + (1-p)\int_0^{\pi} \frac{d\phi}{\pi}\left[g(-\phi) + a\cot\phi\right]$$ (0 by symmetry)

Integrate over angles:

$$\int \frac{d\theta}{\pi} g(-\theta) = \int \frac{d\theta}{\pi} g(\theta) \quad : \text{ undetermined ??}$$

This null vector precisely corresponds to Number conservation of electrons! Set these integral to zero, they will decouple anyway!

Thus $g(-\theta) = p\left[g(\theta) + a\cot\theta\right]$ $\quad (0<\theta<\pi)$

$$g(\theta) = p\left[g(-\theta) + a\cot\theta\right] = p^2\left[g(\theta) + a\cot\theta\right] + pa\cot\theta$$

$$g(\theta) = \frac{p(1+p)}{1-p^2}\, a\cot\theta = \frac{p}{1-p}\, a\cot\theta$$

$$g(-\theta) = p\, a\cot\theta\left(1 + \frac{p}{1-p}\right) = \frac{p}{1-p}\, a\cot\theta$$

The total current flowing is (w/2)

$$I = \int_{-w/2}^{w/2} dy \int \frac{d^2p}{(2\pi\hbar)^2} \langle \Phi(y) | J_x \rangle$$

$$= -eV_F \int_{-w/2}^{w/2} dy \int_0^{\pi} \frac{r\,d\theta}{\pi} \Big[ \underbrace{\cos\theta\left(g(\theta) - eE_x \cot\theta\left(y + \tfrac{w}{2}\right)\right)}_{0<\theta<\pi} + \underbrace{\cos\theta\left(g(-\theta) - eE_x\cot\theta\left(\tfrac{w}{2} - y\right)\right)}_{-\pi<\theta<0} \Big]$$

$$= a\cos\theta\cot\theta\left[\frac{2p}{1-p} + 1\right]$$

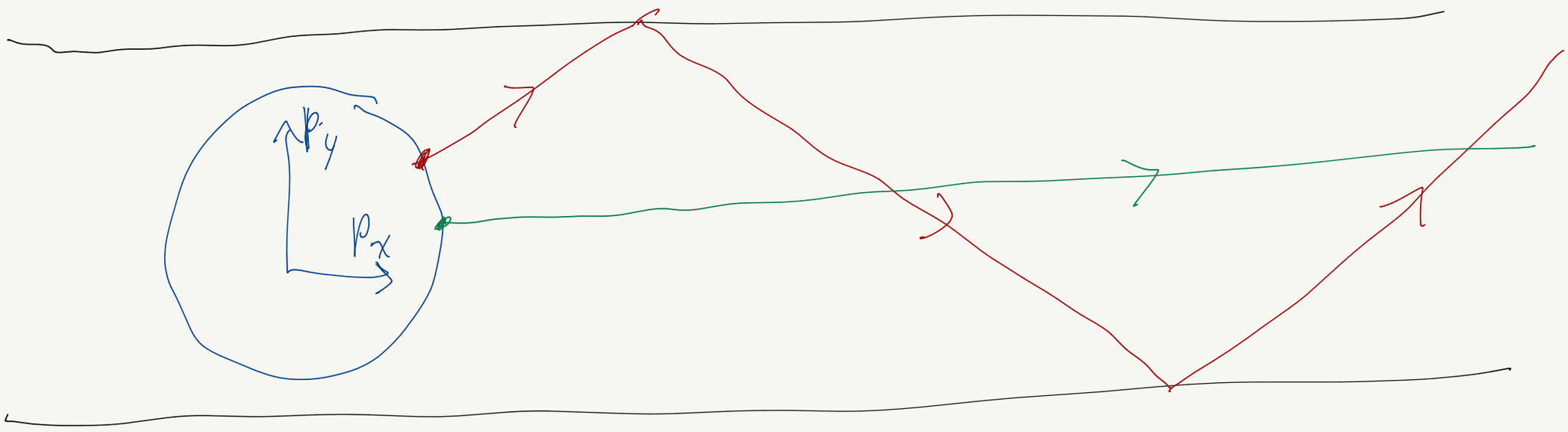
$$\frac{I}{E_x} = e^2 \nu_F w^2 r \int_0^{\pi} \frac{d\theta}{\pi}\, \frac{\cos^2\theta}{\sin\theta}\, \frac{1+p}{1-p}$$

This integral is formally divergent! In a channel with electron electron interactions, however, we could regulate this integral at the scale given by the length of the interactions...

$$\frac{R}{L} = \frac{1}{r\nu_F w^2 e^2}\, \frac{1-p}{1+p} \left( \int_{w/l_{ee}}^{\pi} \frac{d\theta}{\pi}\, \frac{\cos^2\theta}{\sin\theta} \right)^{-1} \sim \frac{1}{r\nu_F^2 w^2 e^2}\, \frac{1-p}{1+p} \times \left( \frac{1}{\pi} \log \frac{l_{ee}}{w} \right)^{-1}$$

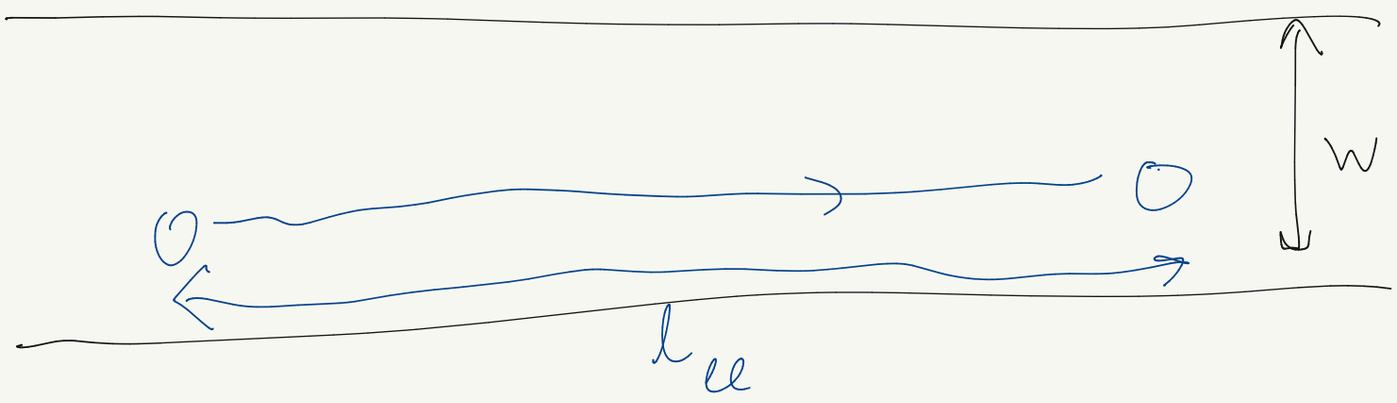
↑ resistance per unit length

To understand the origin of this divergence, let's think about the following




Particles whose velocity is oriented in the x direction traveling down the center of the channel are not easily scattered by the walls! They are also highly accelerated by the electric field since they are aligned by it, and they lead to the logarithmic divergence.

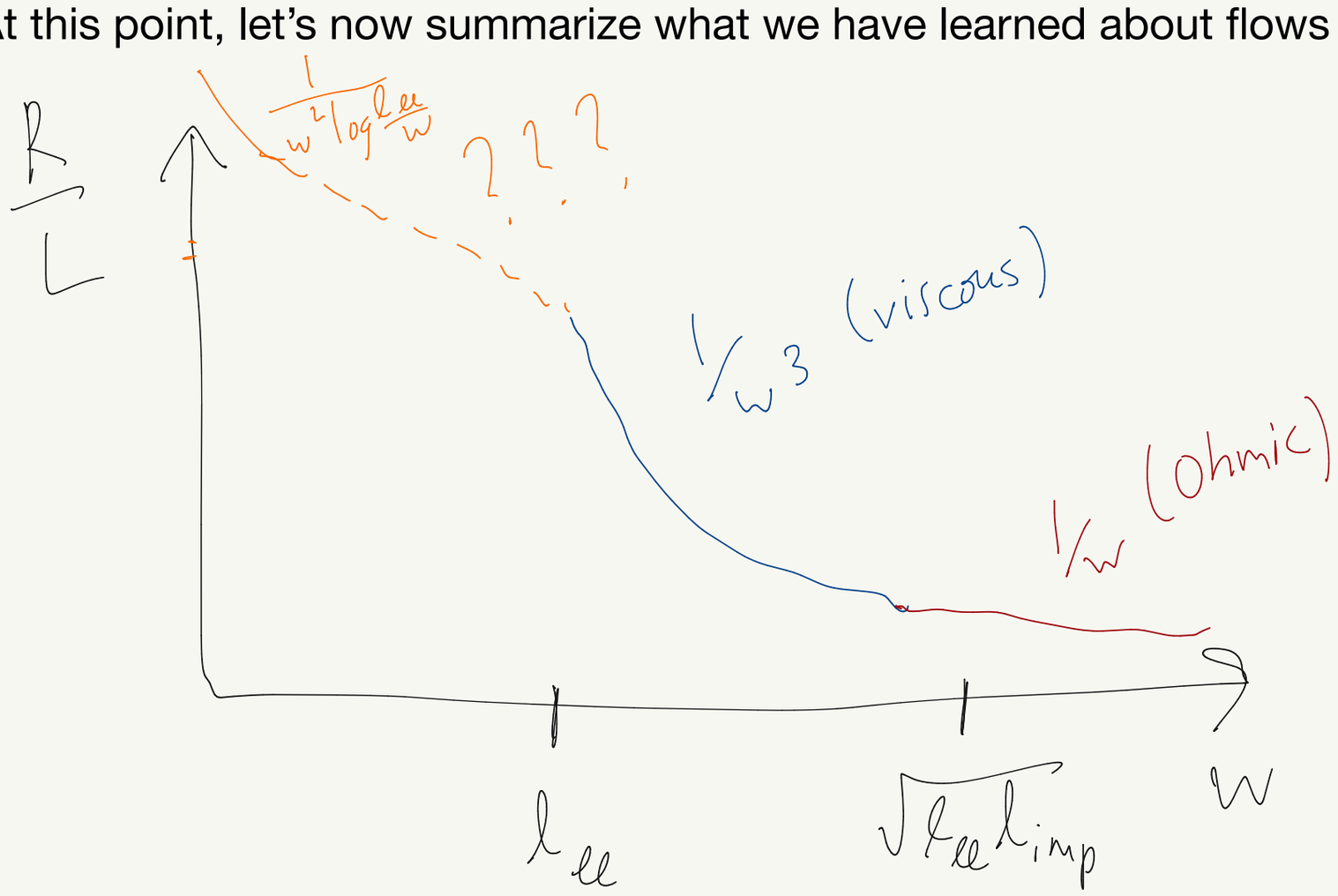
If we have some very weak electron electron scattering, however, we can't accelerate them forever. Eventually after a collision mean free path, they will scatter. So we should only consider angles in the ballistic integral large enough that an electron electron collision can't occur before the boundary scattering...




$$\theta \gtrsim \frac{w}{l_{ee}}$$

The second divergence that we should understand is the divergence as all scattering becomes specular. This is analogous to the Ohmic limited transport in a channel with momentum conserving boundary conditions because after all, specular reflections conserves momentum in the x direction. So physically, we see that the slip length from before is roughly related to the kinetic nature of boundary scattering !

At this point, let's now summarize what we have learned about flows in channels.



It is tempting to just go ahead and connect the ballistic to the hydrodynamic regime, and assume that the resistance is always increasing faster than the Ohmic prediction as the channel gets narrower. Indeed, this is essentially what happens. To rigorously show this, we would need to solve the full kinetic equations with a collision integral, a task that has to be done on a computer...

### 4.2) Transport in inhomogeneous media

Reference: 1706.04621

Our goal is to now repeat our calculation of transport in an inhomogeneous fluid, but in the full kinetic theory. for simplicity, we're going to focus only on the lowest temperature regimes, when we can neglect any radial deformations of the Fermi surface...

$$\partial_t f + \vec{v}\cdot\nabla_x f + \vec{F}\cdot\nabla_p f = -W[f]$$

$$f = f_{eq} - \frac{\partial f_{eq}}{\partial \varepsilon}\Phi(\vec{x},\vec{p}) \cdots$$

Assume $f$ is independent of time $t$. The external forces take the form of

$$\vec{F} = -e\left(\vec{E} - \nabla \mu_{imp}(\vec{x})\right)$$

(solve for $\Phi$ to leading order in $E$...)

where $\mu_{imp}(\vec{x})$ is a weak impurity potential, as in our hydrodynamic calculation.

In equilibrium, $f_{eq} = f_F\left(\varepsilon(\vec{p}) - e\mu_{imp}(\vec{x})\right)$:

$$\left(\vec{v}\cdot\nabla_x + \vec{F}\cdot\nabla_p\right) f_{eq} = \frac{\partial f_F}{\partial \varepsilon}\left(\frac{\partial \varepsilon}{\partial \vec{p}}\cdot\left(-e\frac{\partial \mu_{imp}}{\partial \vec{x}}\right) + e\frac{\partial \mu_{imp}}{\partial x}\cdot\frac{\partial \varepsilon}{\partial p}\right) = 0.$$

To linear order in the electric field: (using our kinetic theory framework)

$$\vec{v}\cdot\nabla_x|\Phi\rangle + e\nabla_x\mu_{imp}\cdot\nabla_p|\Phi\rangle + W|\Phi\rangle = E_i|J_i\rangle$$

where $|\Phi\rangle = \int d^d\vec{p}\ \Phi(\vec{p},x)|\vec{p}\rangle$ $\qquad \langle\vec{p}|\vec{p}'\rangle = \frac{\delta(\vec{p}-\vec{p}')}{(2\pi\hbar)^d}\left(-\frac{\partial f_F}{\partial \varepsilon}\right)_{\vec{p}}$

$$|J_x\rangle = \int d^d\vec{p}\ \left(-e v_x(\vec{p})\right)|\vec{p}\rangle \qquad |P_x\rangle = \int d^d\vec{p}\ p_x|\vec{p}\rangle$$ ← useful momentarily

The method of analysis is the same as in the hydro case. If there were no inhomogeneity....

$$|\Phi\rangle = W^{-1} E_x |J_x\rangle \; ?$$

Only if $W^{-1}|J_x\rangle$ exists.

In our old theory of transport, W always had an inverse. Even with a small Fermi surface, there was always electron-impurity scattering. What's different in this calculation is that we have not yet integrated out the impurities! So it's as if we have to go all the way back and calculate the electron-impurity collision integral from first principles...now in the presence of electron-electron scattering !

If there was no $\mu_{imp}$ ... and $\boxed{W|P_x\rangle = 0}$ $\quad (\langle P_x | J_x \rangle = -en \neq 0)$

$|\Phi\rangle = E_x W^{-1} |J_x\rangle$ diverges!

The electric field just keeps accelerating the charged electron fluid forever! Just like before, we look for a singular solution in the strength of the disorder ...

$$|\Phi\rangle \approx \underbrace{A|P_x\rangle}_{\sim \frac{1}{\mu_{imp}^2}} + \underbrace{|\varphi(\vec{x})\rangle}_{\sim \frac{1}{\mu_{imp}}} + \underbrace{|\tilde{\varphi}(\vec{x})\rangle}_{\sim \mu_{imp}^0} + \cdots$$

At $O\left(\frac{1}{\mu_{imp}}\right)$:

$$\vec{v}\cdot\nabla_x |\varphi\rangle + e\left(\nabla_x \mu_{imp}\right)\cdot\frac{\partial}{\partial \vec{p}}|P_x\rangle A + W|\varphi\rangle = 0.$$

Take the spatial Fourier transform:

$$\underbrace{\left(W + i\vec{k}\cdot\vec{v}\right)}_{G^{-1}} |\varphi(\vec{k})\rangle = A \times \left(-e\, i k_x \mu_{imp}(\vec{k})\right)|n\rangle, \quad \text{where}$$

$$|n\rangle = \frac{\partial}{\partial p_x}|P_x\rangle = \int d^d\vec{p}\;\; |\vec{p}\rangle$$

$$|\varphi(\vec{k})\rangle = A \times (-e)\, ik_x \mu_{imp}(\vec{k})\, G(k)\,|n\rangle$$

Integrate the Boltzmann equation at $O(\mu_{imp}^0)$ over all space...

(boundary terms negligible)

$$\int d^d x \left[ \vec{v}\cdot\nabla_x|\tilde{\varphi}\rangle + W|\tilde{\varphi}\rangle + e\nabla_x\mu_{imp}\cdot\nabla_p|\varphi\rangle \right] = \int d^d x\; E_x |J_x\rangle$$

Apply $\langle P_x|$: $\langle P_x|W = 0$, so...

$$\int d^d x\; e\nabla_x\mu_{imp}\langle P_x|\nabla_p|\varphi\rangle$$

integrate by parts

$$= -e\int d^d x\; \nabla_x\mu_{imp}\langle n|\varphi(x)\rangle = \int d^d x\; E_x\langle P_x|J_x\rangle$$

If we take assume that $|\mu_{imp}(k)|^2 = \underbrace{N_{imp}}_{\text{total \# of impurities}} \underbrace{|U(k)|^2}_{\text{contribution from each impurity}}$:

$$-e\int d^d x\; \nabla_x\mu_{imp}\langle n|\varphi(x)\rangle = \int d^d k\; (-ik_x\mu_{imp}(-k))\langle n|\varphi(k)\rangle(-e)$$

$$= \int d^d k\; k_x^2 |\mu_{imp}(k)|^2 \langle n|G(k)|n\rangle\, e^2 A =$$

Define $\Gamma_{xx} = n_{imp}\int d^d k\; |U(k)|^2 k_x^2 \langle n|G(k)|n\rangle$. Then

impurity density $= \dfrac{N_{imp}}{\int d^d x\, 1}$

$$E_x\langle P_x|J_x\rangle = -enE_x = A\Gamma_{xx}$$

Electrical conductivity given by

$$\sigma_{xx} = \frac{\int d^d x \langle \Phi | J_x \rangle}{\int d^d x \; E_x} = \boxed{\frac{e^2 n^2}{\Gamma_{xx}}}$$

It remains to calculate the coefficient Gamma. At this point, we have to turn to a simplified model of kinetic theory. Let's use our simple model of the circular Fermi surface, wit

$$|\Phi\rangle \approx \int d^2 p \; |\vec{p}\rangle \left( \sum a_m e^{im\theta} \right) \quad , \text{ where } \tan\theta = \frac{p_y}{p_x} .$$

$$= \sum_m a_m |m\rangle .$$

$$\langle m | m' \rangle = \underbrace{2 \delta_{mm'}}$$

can usually be sloppy with this...

$$W|m\rangle = \begin{cases} \gamma & |m| \geq 2 \\ 0 & |m| \leq 1 \end{cases} \longleftarrow \text{charge \& momentum conservation}$$

Charge density $|n\rangle = |0\rangle$. We need to evaluate

$$\langle 0 | (W + i\vec{k}\cdot\vec{v})^{-1} | 0 \rangle$$

Without loss of generality, set $\vec{k} = k_x \hat{x}$. We find

$$W + i\vec{k}\cdot\vec{v} = \begin{pmatrix} \gamma & \frac{ikv_F}{2} & 0 & & & & \\ \frac{ikv_F}{2} & 0 & \frac{ikv_F}{2} & 0 & & & \\ 0 & \frac{ikv_F}{2} & 0 & \frac{ikv_F}{2} & 0 & & \\ & 0 & \frac{ikv_F}{2} & 0 & \frac{ikv_F}{2} & 0 & \\ & & 0 & \frac{ikv_F}{2} & 0 & \frac{ikv_F}{2} & \\ & & & 0 & \frac{ikv_F}{2} & \gamma & \frac{ikv_F}{2} \end{pmatrix}$$

$|-2\rangle \quad |-1\rangle \quad |0\rangle \quad |1\rangle \quad |2\rangle$

To invert this infinite dimensional matrix, we use the following trick. First, a block matrix inversion identity

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & \cdots \\ \cdots & \cdots \end{pmatrix}$$

($A$: $m \times m$, $B$: $m \times n$, $C$: $n \times m$, $D$: $n \times n$)

only need this block...

We now carry out the block matrix inversion in stages. Our goal is to integrate out the modes with m 2 or bigger. But observe a recursive structure to the higher m modes....

$$\left(\begin{array}{c|cccc} |m\rangle & |m+1\rangle & |m+2\rangle & & \\ \hline \gamma & a & 0 & & \\ a & \gamma & a & 0 & \\ 0 & a & \gamma & a & \ddots \\ & 0 & a & \gamma & \ddots \\ & & \ddots & \ddots & \ddots \end{array}\right) = X \qquad \left(a = \frac{ikv_F}{2}\right)$$

this matrix looks like $X$!

Top left component: $\langle m|X^{-1}|m\rangle = \dfrac{1}{\gamma - a^2\langle m+1|X^{-1}|m+1\rangle}$

$C$: $C(\gamma - a^2 C) = 1$

$$\Rightarrow C = \frac{\gamma \pm \sqrt{\gamma - 4a^2}}{2a^2} \rightarrow \frac{\gamma - \sqrt{\gamma^2 + k^2 v_F^2}}{-\frac{k^2 v_F^2}{2}}$$

We took the minus sign b/c if $k=0$, $C = \frac{1}{\gamma}$ ...

Defining $\gamma(k) = \frac{1}{2}\left(\sqrt{\gamma^2 + (kv_F)^2} - \gamma\right)$, we conclude that

$$\langle 0|G|0\rangle = \langle 0| \begin{pmatrix} \gamma(k) & \frac{ikv_F}{2} & 0 \\ \frac{ikv_F}{2} & 0 & ikv_F/2 \\ 0 & \frac{ikv_F}{2} & \gamma(k) \end{pmatrix}^{-1} |0\rangle$$

$|-1\rangle \quad |0\rangle \quad |1\rangle$

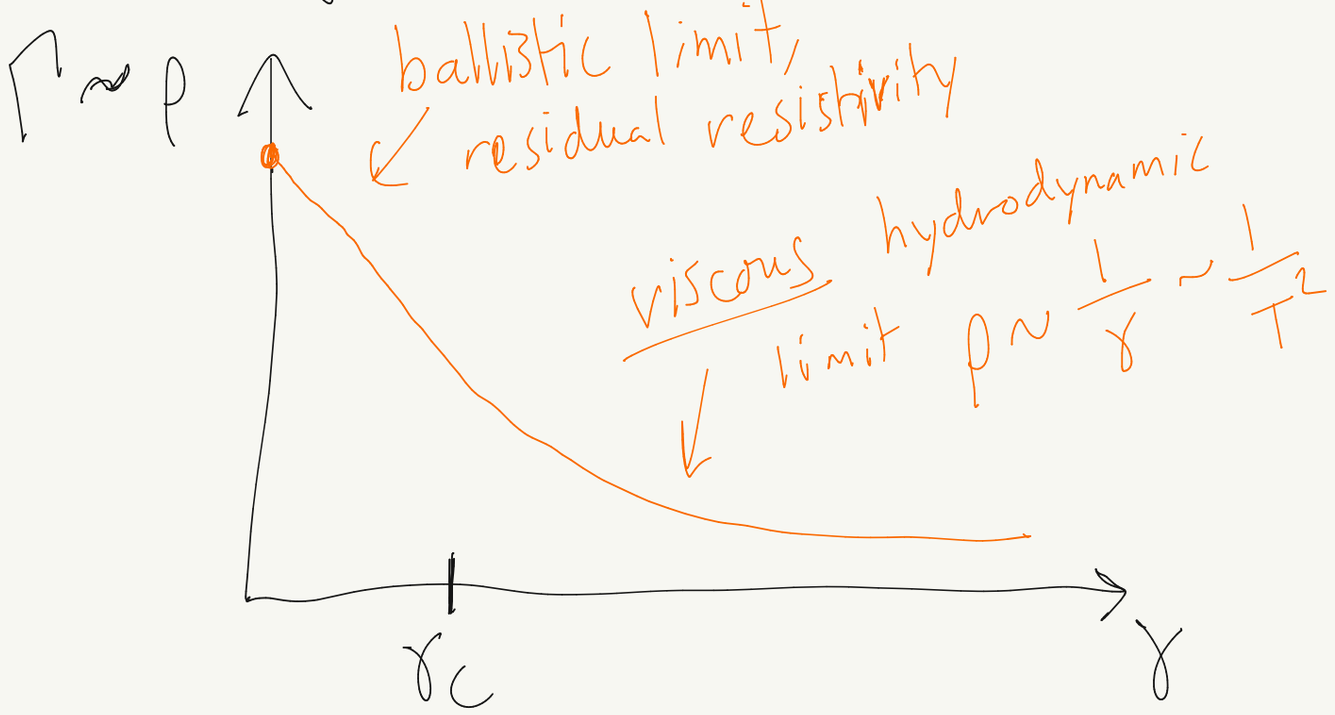
Change of basis to $|X\rangle = \frac{|1\rangle + |-1\rangle}{\sqrt{2}}$, $|Y\rangle = \frac{|1\rangle - |-1\rangle}{\sqrt{2}}$ ...

$$\langle 0|G|0\rangle = \langle 0| \begin{pmatrix} 0 & ikv_F/\sqrt{2} & 0 \\ ikv_F/\sqrt{2} & \gamma(k) & 0 \\ 0 & 0 & \gamma(k) \end{pmatrix}^{-1} |0\rangle$$

$|0\rangle \quad |X\rangle \quad |Y\rangle$

$$\frac{\langle 0|G|0\rangle}{\langle 0|0\rangle} = \frac{2}{k^2 v_F^2}\gamma(k) = \frac{\sqrt{\gamma^2 + (kv_F)^2} - \gamma}{k^2 v_F^2} = \frac{1}{\sqrt{\gamma^2 + (kv_F)^2} + \gamma}$$

This is a decreasing function of $\gamma$: the momentum conserving scattering enhances transport & decreases resistivity.




This is a bulk manifestation of the Gurzhi effect!

crossover scale $\gamma_c \sim \frac{v_F}{\xi}$, where $\xi$ is length scale of disorder

Let's check that we have correctly reproduced the hydrodynamic limit, compared to what we found before...

hydro regime: $\Gamma_{xx} = \int d^d k \; \eta \left(\frac{\nu}{n}\right)^2 k_x^2 \; |\mu_{imp}(k)|^2$

$\nu$ = density of states $= \frac{\partial n}{\partial \mu}$, $n$ = equilibrium density

For the circular Fermi surface...

$$\frac{\nu}{n} = \frac{2}{p_F v_F}, \quad \eta = \left(\frac{p_F v_F}{2}\right)^2 \frac{\nu}{2\gamma}, \quad \text{so}$$

$$\Gamma_{xx} = \int d^d k \; k_x^2 |\mu_{imp}(k)|^2 \frac{\nu}{2\gamma} \approx \int d^d k \; k_x^2 |\mu_{imp}|^2 \langle 0|G|0\rangle$$

since $\langle 0|0\rangle = \nu$ and $\frac{\langle 0|G|0\rangle}{\langle 0|0\rangle} = \frac{1}{2\gamma}$ when $\gamma \gg \gamma_c$!

Thus we see that in this simplified model, the momentum conserving interactions decrease the resistivity further. This effect has never been convincingly seen in bulk resistance but has been seen in more tailored setups...

### 4.3) Zero-to-first sound crossover

Reference: 1801.01495

While we are solving this simplified model of a kinetic theory for a two dimensional Fermi liquid, we ought to also revisit the propagation of the normal modes. Consider the plane wave solutions of the unsourced kinetic equation:

$$\partial_t |\Phi\rangle + W|\Phi\rangle + \vec{v}\cdot\nabla_x|\Phi\rangle = 0$$

$$-i\omega|\Phi\rangle + W|\Phi\rangle + i\vec{k}\cdot\vec{v}|\Phi\rangle = 0$$

Let's ask what happens to the hydrodynamic modes as we begin to approach time and length scales which are comparable to or shorter than the scattering time. To do this, we use a trick

the matrix $M = (-i\omega + W + i\vec{k}\cdot\vec{v})^{-1}$ has a <u>pole</u> (diverges) when $(\omega, \vec{k})$ are associated w/ a quasinormal mode. For simplicity let's focus on the $3\times3$ block of $M$ is the $|m|\leq 1$ harmonic sector. Why? Well we can basically just use our earlier results! Take $\vec{k} = k\hat{x}$ as before...

$$M = \begin{pmatrix} \gamma - i\omega & \frac{ikv_F}{2} & 0 \\ \frac{ikv_F}{2} & -i\omega & \frac{ikv_F}{2} \\ 0 & \frac{ikv_F}{2} & \gamma - i\omega \\ |-1\rangle & |0\rangle & |1\rangle \end{pmatrix}^{-1}$$

where $\gamma = \frac{\sqrt{(\gamma - i\omega)^2 + (kv_F)^2} - \gamma + i\omega}{2}$

Do the same change of basis:

$$|X\rangle = \frac{|1\rangle + |-1\rangle}{\sqrt{2}}$$

$$|Y\rangle = \frac{|1\rangle - |-1\rangle}{\sqrt{2}}$$

$$M = \begin{pmatrix} -i\omega & \frac{ikv_F}{\sqrt{2}} & 0 \\ \frac{ikv_F}{\sqrt{2}} & \gamma - i\omega & 0 \\ 0 & 0 & \gamma - i\omega \\ |0\rangle & |X\rangle & |Y\rangle \end{pmatrix}$$

The diffusive transverse momentum mode:

$$i\omega = \gamma = \frac{i\omega}{2} - \frac{\gamma}{2} + \frac{\sqrt{(\gamma - i\omega)^2 + (kv_F)^2}}{2}$$

$$(i\omega + \gamma)^2 = (\gamma - i\omega)^2 + (kv_F)^2 \Rightarrow 0 = -4\gamma i\omega + (kv_F)^2$$

$$\omega = -i\frac{v_F^2}{4\gamma}k^2$$ (same diffusion constant as before... curiosity of this toy model!)

For the sound waves, we instead find that

$$0 = -i\omega(\gamma - i\omega) + \frac{k^2 v_F^2}{2}$$

To analyze this equation...

When $\gamma \gg \omega$: $0 = \frac{k^2 v_F^2}{2} - \omega^2 - i\omega \frac{k^2 v_F^2}{4(\gamma - i\omega)}$

$$\omega \simeq \frac{-i\frac{k^2 v_F^2}{4\gamma} \pm \sqrt{2k^2 v_F^2}}{2} = \pm \frac{k v_F}{\sqrt{2}} - \frac{i k^2 v_F^2}{8\gamma} + \cdots$$

This is the ordinary sound wave. At shorter time scales, we instead have $(\gamma \ll \omega)$

$$0 = \frac{k^2 v_F^2}{2} - \omega^2 - i\omega \frac{\sqrt{(\gamma - i\omega)^2 + (k v_F)^2} + (\gamma - i\omega)}{2}$$

$$= \frac{k^2 v_F^2 - \omega^2 + i\omega\gamma - i\omega\sqrt{k^2 v_F^2 - \omega^2 - 2i\omega\gamma + \gamma^2}}{2}$$

As $\gamma \to 0$, we have $\omega = \pm k v_F$. Let's try $\omega = \pm k v_F + \underbrace{\int}_{\mathcal{O}(k^0)} + \cdots$

Plug in $\omega = k v_F + \int + \cdots$: at $\mathcal{O}(k)$:

$$0 = -2 k v_F \int + i k v_F \gamma - i k v_F \sqrt{\gamma^2 - 2 i k v_F \gamma - 2 k v_F \int}\ ?$$

Satisfied if $\int = -i\gamma$! Hence we find: