5. Classical magnetotransport

5.1) The Hall effect

We now talk about transport phenomena in the presence of a background magnetic field. Let's begin with our Drude model...

Lorentz force

$$\underbrace{\frac{P_i}{\tau}}_{\text{momentum relaxation}} = \rho E_i + (J \times B)_i$$

$$= \underbrace{\rho}_{\text{charge density}} E_i + B_{ij} J_j \,, \quad \text{where the matrix } B = \begin{pmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{pmatrix}$$

Before solving this, let's take a step back. Suppose momentum relaxation is negligible. Then we actually get a very simple formula for resistivity!

$$E_i = \underbrace{-\frac{1}{\rho} B_{ij}}_{\text{resistivity tensor!}} J_j$$

$$\boxed{(\sigma_{ij})^{-1} = -\frac{1}{\rho} B_{ij}}$$

This formula which holds in the absence of momentum relaxation is extremely universal. This is often called the classical Hall effect, but this formula would hold in the quantum Hall effect also! It's a simple requirement from momentum conservation!

One thing that might seem puzzling is that the conductivity/resistivity tensors are antisymmetric. This is not a violation of Onsager reciprocity because the magnetic field breaks time reversal invariance!

The Hall effect is a very useful way of measuring the density of charge carriers in a metal.

Lastly let's still check that the Hall conductivity is positive semi definite

test vector ϕ_i:

$$\phi_i (\sigma_{ij})^{-1} \phi_j = -\frac{1}{\rho} \underbrace{\phi_i B_{ij} \phi_j}_{\text{since } B_{ij} \text{ is antisymmetric!}} = 0$$

5.2) Absence of Drude magnetoresistance in isotropic metals

Now let's add in momentum relaxation and solve our Drude model.

As in our original Drude model, let's approximate $J_i = -\frac{e}{m} P_i$.

$$-\frac{mJ_i}{e\tau} = -\underbrace{en}_{\text{charge density}} E_i + B_{ij} J_j$$

$$\underbrace{\left(\frac{m}{ne^2\tau}\delta_{ij} + \frac{1}{en}B_{ij}\right)}_{=\rho_{ij} = \text{resistivity tensor.}} J_j = E_i$$

For example let's take magnetic field to point in the z-direction...

$$\rho = \begin{pmatrix} \rho_{xx} & \rho_{xy} & \rho_{xz} \\ \rho_{yx} & \rho_{yy} & \rho_{yz} \\ \rho_{zx} & \rho_{zy} & \rho_{zz} \end{pmatrix} = \begin{pmatrix} \rho_0 & \frac{B}{en} & 0 \\ -\frac{B}{en} & \rho_0 & 0 \\ 0 & 0 & \rho_0 \end{pmatrix},$$

where $\rho_0 = \frac{m}{ne^2\tau}$ was ordinary Drude resistance!

Hence our simple Drude model of transport predicts that the dissipative xx-components of resistivity do not care about the magnetic field!

magnetoresistance $\frac{\partial \rho_{xx}}{\partial (B^2)} = 0!$

In contrast, if we look at conductivity... components do depend on B-field!

defining cyclotron frequency

$$\omega_c = \frac{eB}{m} \quad \left(\rho_{xy}/\rho_{xx} = \omega_c\tau\right) \implies \sigma = \rho^{-1} = \frac{1}{\rho_0}\begin{pmatrix} \frac{1}{1+(\omega_c\tau)^2} & \frac{\omega_c\tau}{1+(\omega_c\tau)^2} & 0 \\ \frac{-\omega_c\tau}{1+(\omega_c\tau)^2} & \frac{1}{1+(\omega_c\tau)^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

5.3) Kinetic theory of magnetotransport

That very simple Drude cartoon of magnetotransport is obviously too simplistic. So now let's begin by adding magnetic field effects back in to our kinetic theory of transport...

Recall: distribution function $f(x,p) = f_{eq}(x,p) - \frac{\partial f_{eq}}{\partial \varepsilon} \Phi + \cdots$
linear in E and $-\nabla T$

$$|\Phi\rangle = \int d^d p \; \Phi(\vec{p})\, |\vec{p}\rangle, \quad \text{with} \quad \langle \vec{p}|\vec{p}\,'\rangle = \frac{1}{(2\pi\hbar)^d}\left(-\frac{\partial f_{eq}}{\partial \varepsilon}\right)\delta(\vec{p}-\vec{p}\,').$$

Let's focus on a fermionic theory of the electrons in a metal.
The Boltzmann equation reads [assuming spatial homogeneity $\Rightarrow \nabla_x = 0$]

$$\partial_t|\Phi\rangle + \vec{F}_{mag}\cdot\nabla_p|\Phi\rangle + W|\Phi\rangle = E_i|J_i\rangle$$

where $|J_i\rangle = -e\int d^d p \; v_i(\vec{p})|\vec{p}\rangle$ and W is the collision integral from before!

$\vec{F}_{mag}$ represents the magnetic Lorentz force on a quasiparticle:

$$(F_{mag})_i = -e B_{ij} v_j(\vec{p})$$

Let's define $W_{mag}|\Phi\rangle = \vec{F}_{mag}\cdot\nabla_p|\Phi\rangle$. Note

$$\langle \Phi_1|W_{mag}|\Phi_2\rangle = \int \frac{d^d p}{(2\pi\hbar)^d}\left(-\frac{\partial f_F}{\partial \varepsilon}\right)_p \left(-eB_{ij}v_j(p)\right)\Phi_1 \frac{\partial \Phi_2}{\partial p_i}$$

$$= -\int \frac{d^d p}{(2\pi\hbar)^d}\left(-\frac{\partial f_F}{\partial \varepsilon}\right)_p \left(-eB_{ij}v_j(p)\right)\Phi_2 \frac{\partial \Phi_1}{\partial p_i}$$

$$- \int \frac{d^d p}{(2\pi\hbar)^d}\Phi_2\Phi_1 \underbrace{\frac{\partial}{\partial p_i}\left[\frac{\partial}{\partial p_j}\left(-e B_{ij} f_F\right)\right]}_{=0 \text{ since } B_{ij} \text{ is antisymmetric!}}$$

$$= -\langle \Phi_2|W_{mag}|\Phi_1\rangle$$

The electrical conductivity tensor is given by

$$\sigma_{ij} = \langle J_i | (W + W_{mag})^{-1} | J_j \rangle$$

The thermoelectric conductivities are similar:

$$T\alpha_{ij} = \langle J_i | (W + W_{mag})^{-1} | Q_j \rangle \qquad T\bar{\kappa}_{ij} = \langle Q_i | (W + W_{mag})^{-1} | Q_j \rangle.$$

Experimentalist's thermal conductivity:

$$\kappa_{ij} = \bar{\kappa}_{ij} - T\alpha_{ki} (\sigma^{-1})_{kl} \alpha_{lj}$$

As a simple example, let's evaluate the conductivity using a relaxation time approximation for the ordinary part of the collision integral:

$$W|\Phi\rangle \approx \frac{1}{\tau}|\Phi\rangle.$$

We evaluate the matrix inverse using a trick....

for matrix M: $\quad M^{-1} = \int_{-\infty}^{0} ds\, e^{sM}$

Now, $e^{sW_{mag}}|\Phi\rangle = |\hat{\Phi}(s)\rangle$, where $\partial_s |\hat{\Phi}\rangle = W_{mag}|\hat{\Phi}\rangle$

$$= \vec{F}_{mag} \cdot \nabla_p |\hat{\Phi}\rangle$$

In other words, $\partial_s \hat{\Phi}(\vec{p}, s) + \vec{F}_{mag} \cdot \nabla_p \hat{\Phi}(\vec{p}, s) = 0 \ldots$ this is just a Liouville equation in a magnetic field, so we conclude that

$$\hat{\Phi}(\vec{p}, 0) = \hat{\Phi}(\vec{p}(s), s), \quad \text{where} \quad \boxed{\partial_s \vec{p} = \vec{F}_{mag}}$$

Hence we arrive at Chambers' formula:

$$\sigma_{ij} = \int_{-\infty}^{0} ds \int \frac{d^d p}{(2\pi\hbar)^d} \left(-\frac{\partial f_E}{\partial \varepsilon}\right)_p v_i(p)\, e^{s/\tau} v_j(p(s))\, e^2$$

Let's evaluate this for a circular Fermi surface

$$\sigma_{xy} = \frac{e^2 p_F}{(2\pi\hbar)^2 v_F} \int_0^{2\pi} d\theta \; v_F \cos\theta \int_{-\infty}^{0} ds \, e^{s/\tau} \sin(\theta - \omega_c s) v_F$$

Hall conductivity

$$= \frac{p_F v_F e^2}{(2\pi\hbar)^2} \int_0^{2\pi} d\theta \int_{-\infty}^{0} ds \, e^{s/\tau} \, \frac{\sin(2\theta - \omega_c s) - \sin(\omega_c s)}{2}$$

$$= \frac{p_F v_F e^2}{4\pi\hbar^2} \int_{-\infty}^{0} ds \; \mathrm{Im}\left(e^{s\left(\frac{1}{\tau} - i\omega_c\right)}\right)$$

$$= \frac{p_F v_F e^2}{4\pi\hbar^2} \mathrm{Im}\left(\frac{1}{\frac{1}{\tau} - i\omega_c}\right) = \frac{\nu v_F^2 e^2 \tau}{2} \frac{\omega_c \tau}{1 + (\omega_c \tau)^2} \quad \checkmark$$

$$\sigma_{xx} = \frac{e^2 p_F}{(2\pi\hbar)^2 v_F} \int_0^{2\pi} d\theta \int_{-\infty}^{0} ds \; e^{s/\tau} v_F^2 \cos\theta \cos(\theta - \omega_c s) = \frac{e^2 p_F v_F}{(2\pi\hbar)^2} \int_0^{2\pi} d\theta \int_{-\infty}^{0} ds \, e^{s/\tau} \frac{\cos(2\theta - \omega_c s) +}{2}$$

$$= \frac{e^2 p_F v_F}{(2\pi\hbar)^2} \int_0^{2\pi} d\theta \int_{-\infty}^{0} ds \; e^{s/\tau} \frac{\cos(2\theta - \omega_c s) + \cos(\omega_c s)}{2} = \frac{\nu}{2} e^2 v_F^2 \int_{-\infty}^{0} ds \; \mathrm{Re}\left(e^{s\left(\frac{1}{\tau} - i\omega_c\right)}\right)$$

$$= \frac{\nu e^2 v_F^2 \tau}{2} \frac{1}{1 + (\omega_c \tau)^2} \quad \checkmark$$

In our kinetic theory, we can always show that the magnetic contribution to transport is not "dissipative" in that it does not contribute to entropy production...generalizing the derivation from before, we find that

$$T\dot{s} = \langle \Phi | (W + W_{mag}) | \Phi \rangle = \langle \Phi | W | \Phi \rangle$$

since W_{mag} is antisymmetric.

Unfortunately our variational principle for transport no longer holds! So magnetic fields can both contribute to dissipationless transport coefficients such as Hall conductivity, as well as modify dissipative coefficients...

5.4) Incoherent conductivity and magnetotransport

As an explicit example of kinetic theory with magnetotransport, let us imagine a toy model of a metal where not all of the current is proportional to momentum. We assume two spatial dimensions

Suppose:

$$W = \underbrace{\gamma\left(1 - \frac{|P_x)(P_x|}{(P_x|P_x)} - \frac{|P_y)(P_y|}{(P_y|P_y)}\right)}_{\text{momentum not relaxed by electron-electron collisions}} + \underbrace{\Gamma}_{\text{"everything" scatters off impurities... a bit lazy!}}$$

We make an uncontrolled approximation that the only relevant vectors (I.e. kinds of perturbations to the distribution function) are the two components of the momentum and the current.

$$|P_x), |P_y), |J_x), |J_y).$$

Orthogonal basis has incoherent part of currents:

$$|\tilde{J}_i) = |J_i\rangle - \frac{\langle P_x|J_x\rangle}{\langle P_x|P_x\rangle}|P_i)$$

Now observe that

$$W_{mag}|P_k\rangle = \int d^2\vec{p}\left(-eB\,\varepsilon_{ij}v_j(\vec{p})\frac{\partial}{\partial p_i}\right)p_k|\vec{p}\rangle$$

$$= \int d^2\vec{p}\,(-eB)\,\varepsilon_{kj}v_j(\vec{p})|\vec{p})$$

$$= B\,\varepsilon_{kj}|J_j\rangle$$

let's define/assume

$$\langle P_x|P_x\rangle = \langle P_y|P_y\rangle = \mathcal{M}$$

$$\langle \tilde{J}_x|\tilde{J}_x\rangle = \langle \tilde{J}_y|\tilde{J}_y\rangle = C.$$

Recall that

$$\langle P_x|J_x\rangle = \langle P_y|J_y\rangle = -en.$$

Lastly, let's assume that

$$\langle \tilde{J}_i|W_{mag}|\tilde{J}_j\rangle = \underbrace{D}_{\text{unknown constant}}\varepsilon_{ij}$$

So now we write out our dissipative and magnetic collision integrals...

$$
W = \begin{array}{c} \begin{array}{cccc} |P_x\rangle & |P_y\rangle & |\tilde{J}_x\rangle & |\tilde{J}_y\rangle \end{array} \\ \begin{pmatrix} \Gamma & 0 & 0 & 0 \\ 0 & \Gamma & 0 & 0 \\ 0 & 0 & \Gamma+\gamma & 0 \\ 0 & 0 & 0 & \Gamma+\gamma \end{pmatrix} \end{array}
\qquad
W_{mag} = \begin{array}{c} \begin{array}{cccc} |P_x\rangle & |P_y\rangle & |\tilde{J}_x\rangle & |\tilde{J}_y\rangle \end{array} \\ \begin{pmatrix} 0 & \omega_c & 0 & -\Omega_2 \\ -\omega_c & 0 & \Omega_2 & 0 \\ 0 & -\Omega_2 & 0 & \Omega_1 \\ \Omega_2 & 0 & -\Omega_1 & 0 \end{pmatrix} \end{array}
$$

where $\omega_c = \dfrac{eBn}{\mathcal{M}}$ is the cyclotron frequency, and

$\Omega_1 = \dfrac{D}{C}$, $\Omega_2 = B\sqrt{\dfrac{C}{\mathcal{M}}}$. Note that we have normalized

$$\frac{\langle\phi_1|W_{mag}|\phi_2\rangle}{\sqrt{\langle\phi_1|\phi_1\rangle\langle\phi_2|\phi_2\rangle}}$$

above, so that we can more easily take the matrix inverse.

Now we take the matrix inverse of the combined collision integral. The answer is not very enlightening. So let's just write down our final formula for the conductivity tensor...

Define: $\sigma_0 = \dfrac{C}{\gamma} + \dfrac{e\Omega_1 n}{B\gamma}$ to be an incoherent conductivity...

$$\sigma_{xx} = \sigma_{xy} = \frac{(\sigma_0 B)^2 + \mathcal{M}\Gamma\sigma_0 + e^2n^2\left(1 - \frac{\Gamma\Omega_1}{\gamma\omega_c}\right)}{\left(\sigma_0\frac{B^2}{\mathcal{M}} + \Gamma\right)^2 + \left(\omega_c + \frac{\Gamma\Omega_1}{\gamma}\right)^2}\,\frac{\Gamma}{\mathcal{M}}$$

$$\sigma_{xy} = -\sigma_{yx} = \frac{-en\left[\left(\omega_c + \frac{\Gamma\Omega_1}{\gamma}\right)^2 + \left(\sigma_0\frac{B^2}{\mathcal{M}} + \Gamma\right)^2 - \Gamma^2\right] + \frac{\sigma_0 B\Omega_1\Gamma^2}{\gamma}}{\left(\sigma_0\frac{B^2}{\mathcal{M}} + \Gamma\right)^2 + \left(\omega_c + \frac{\Gamma\Omega_1}{\gamma}\right)^2}\,\frac{1}{B}$$

In the limit where momentum relaxation becomes negligible, we see that

$$\sigma_{xx} \to 0, \qquad \sigma_{xy} = \frac{en}{B}$$

If momentum relaxation is finite and the magnetic field vanishes, we obtain that instead

$$\sigma_{xx} = \frac{e^2 n^2}{\mathcal{M}\Gamma} + \left\{ \sigma_0 - \frac{e^2 n^2 \Omega_1}{\gamma \omega_c} \right\} \leftarrow \text{incoherent contribution!}$$

Drude / momentum relaxing contribution

Now let's calculate magnetoresistance:

For simplicity, set $\Omega_1 = 0 \ldots$

$$\rho_{xx} = \frac{\frac{\mathcal{M}\Gamma}{e^2 n^2}\left[1 + \sigma_0 \frac{\Gamma \mathcal{M}}{n^2 e^2} + \left(\frac{\sigma_0 B}{en}\right)^2\right]}{\left(1 + \frac{\Gamma \sigma_0 \mathcal{M}}{n^2 e^2}\right)^2 + \left(\frac{\sigma_0 B}{en}\right)^2}$$

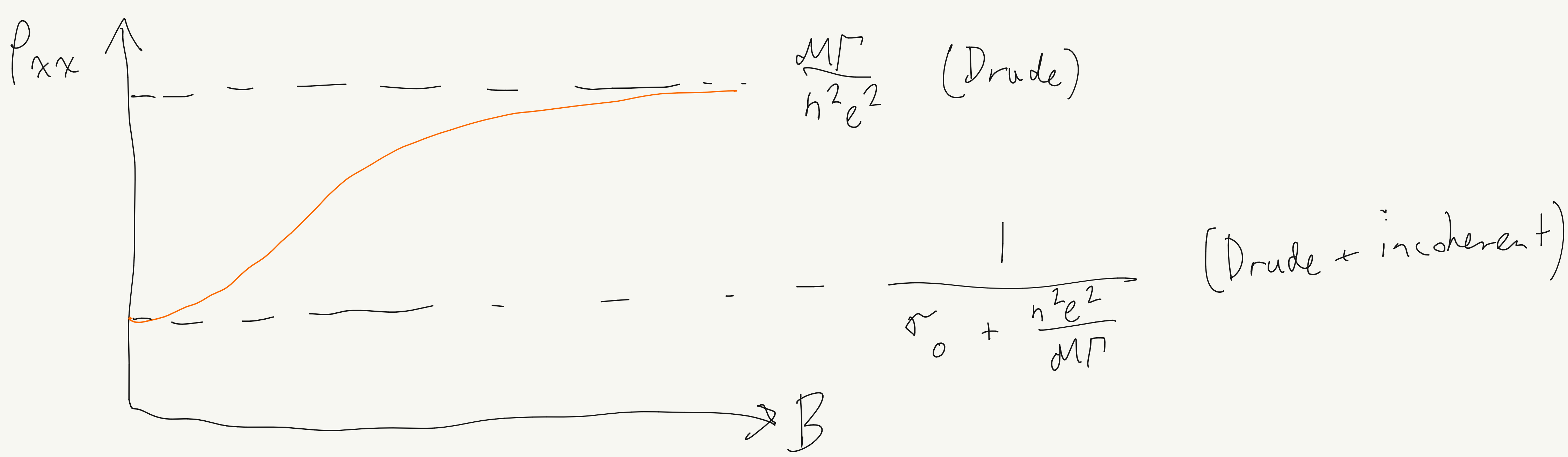
Using the fact that

$$\text{If } a > 1, \quad \frac{a + x}{a^2 + x} \approx \frac{1}{a} + \frac{x}{a^2} - \frac{x}{a^3} + \mathcal{O}(x^2)$$

$$\Rightarrow \boxed{\frac{\partial \rho_{xx}}{\partial B^2} > 0}$$

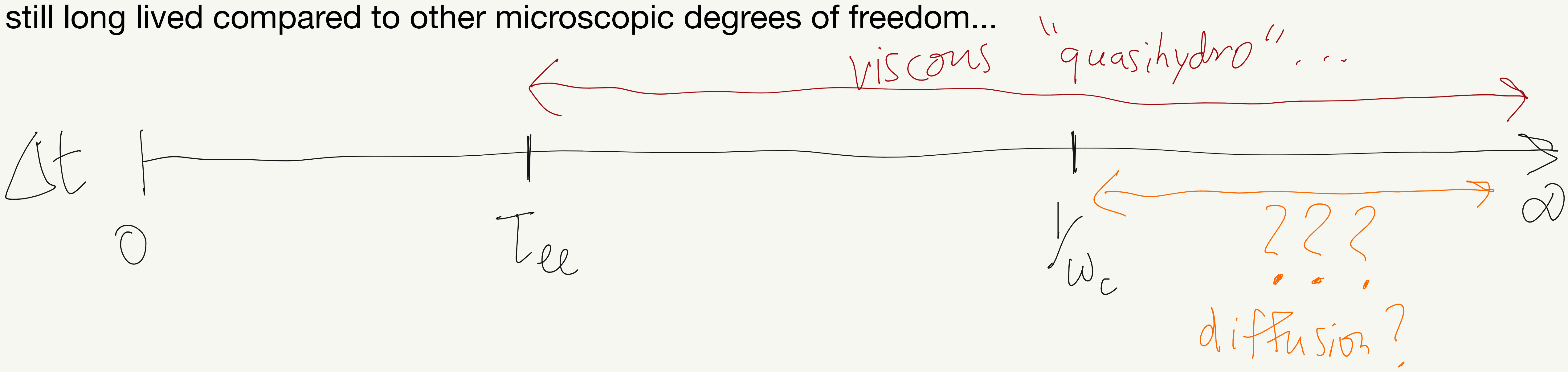
So in general, magnetoresistance is positive in two dimensions for the incoherent metal. The situation can be more complicated in higher dimensions.

Also observe that in general, the incoherent conductivity has decreased the electrical resistivity. Only when the magnetic field is sufficiently large do we recover Drude transport



5.5) Hall viscosity

We now turn to the study of hydrodynamic modes in a background magnetic field. Before beginning, we should emphasize that momentum is no longer conserved in the presence of a magnetic field...so we are really thinking about a "quasihydrodynamic" limit where the magnetic field is small enough that momentum is still long lived compared to other microscopic degrees of freedom...



For simplicity, let's focus on the low temperature limit of a Fermi liquid, where we can approximately ignore energy conservation and focus only on charge and momentum conservation. To first understand the hydrodynamic regime, let's first derive it from kinetic theory. We have

$$\vec{v}\cdot\nabla_x \Phi + \vec{F}_{mag}\cdot\nabla_p \Phi = -W[\Phi]$$

Previously we went to the harmonic basis...for an isotropic system in two spatial dimensions

$$|\Phi\rangle = \sum_{m=-\infty}^{\infty} \Phi_m(\vec{x},t)\underbrace{|m\rangle}_{e^{im\theta}} \qquad \theta = \tan\frac{p_y}{p_x}$$

let's now calculate

$$\langle m'|\vec{F}_{mag}\cdot\nabla_p|m\rangle = \nu\int_0^{2\pi}\frac{d\theta}{2\pi}\, e^{-im'\theta}\left(-eB\varepsilon_{ij}v_j\frac{\partial}{\partial p_i}\right)e^{im\theta}$$

$$= \nu\int_0^{2\pi}\frac{d\theta}{2\pi}\, e^{-im'\theta}\left(\frac{eBv_F}{p_F}\frac{\partial}{\partial\theta}\right)e^{im\theta}$$

$$= \nu\,\omega_c\, i m\int_0^{2\pi}\frac{d\theta}{2\pi}\, e^{i(m-m')\theta} = i m\,\omega_c\,\nu\,\delta_{m,m'}$$

Note that this does not mix different angular harmonics, this is a consequence of rotational invariance.

We can treat this problem much like what we have solved before, with an effective collision integral consisting of momentum conserving interactions along with the magnetic contributions. Let's work in our relaxation time approximation, where we obtain that

$$W_{eff} = \begin{array}{c} \begin{pmatrix} \frac{1}{\tau_{ee}} - 2i\omega_c & 0 & & & \\ 0 & -i\omega_c & 0 & & \\ & 0 & 0 & 0 & \\ & & 0 & +i\omega_c & 0 \\ & & & 0 & \frac{1}{\tau_{ee}} + 2i\omega_c \end{pmatrix} \\ \begin{array}{ccccc} |-2\rangle & \quad |-1\rangle & \quad |0\rangle & \quad |1\rangle & \quad |2\rangle \end{array} \end{array}$$

only non zero on the diagonal

In the hydrodynamic limit, we can directly integrate out the harmonics except for 0, 1 and -1. We obtain that

$$\partial_t \Phi_0 + \frac{v_F}{2}\left(\partial_+ \Phi_{-1} + \partial_- \Phi_1\right) = 0 \qquad \text{where } \partial_x \mp i\partial_y = \partial_\pm$$

$$\partial_t \Phi_{\pm 1} + \frac{v_F}{2}\left(\partial_\pm \Phi_0 + \partial_\mp \Phi_{\pm 2}\right) = \mp i\omega_c \Phi_{\pm 1}$$

$$\frac{v_F}{2}\partial_\pm \Phi_{\pm 1} \approx -\left(\frac{1}{\tau_{ee}} \pm 2i\omega_c\right)\Phi_{\pm 2}$$

$$m = \frac{p_F}{v_F}$$

Since $\langle 0|\Phi\rangle = \delta n$,

$$\langle p_x|\Phi\rangle = \frac{p_F}{2}\left(\langle 1| + \langle -1|\right)|\Phi\rangle = \frac{p_F}{2}\left(\Phi_1 + \Phi_{-1}\right) = mn\,\delta u_x$$

$$\langle p_y|\Phi\rangle = \frac{\langle 1| - \langle -1|}{-2i}\, p_F |\Phi\rangle = \frac{p_F}{2}\left(i\Phi_1 - i\Phi_{-1}\right) = mn\,\delta u_y$$

$$\partial_t \delta n + n\,\partial_x \delta u_x + n\,\partial_y \delta u_y = 0, \quad \text{using } \Phi_0 \text{ EOM.}$$

Using Φ_1 EOM (multiplied by p_F):

$$\partial_t\left(mn(\delta u_x - i\delta u_y)\right) + \frac{p_F v_F}{2}(\partial_x - i\partial_y)\delta n - \frac{v_F^2}{4\left(\frac{1}{\tau_{ee}} + 2i\omega_c\right)}\left(\partial_x^2 + \partial_y^2\right)\left(mn(\delta u_x - i\delta u_y)\right)$$

$$= -i\omega_c\, mn(\delta u_x - i\delta u_y)$$

In index notation:

$$\partial_t \delta u_i + \frac{v_F^2}{2n}\partial_i \delta n - \frac{\eta}{mn}\partial_j\partial_j \delta u_i - \frac{\eta_H}{mn}\varepsilon_{ik}\partial_j\partial_j \delta u_k = -\omega_c \varepsilon_{ik}\delta u_k$$

where $$\underbrace{\eta = mn\frac{v_F^2 \tau_{ee}}{4(1+(2\omega_c\tau_{ee})^2)}}_{\text{shear viscosity}}, \quad \underbrace{\eta_H = -mn\frac{v_F^2\omega_c\tau_{ee}^2}{2(1+(2\omega_c\tau_{ee})^2)}}_{\text{Hall viscosity.}}$$

Strictly speaking, we should only take these formulas seriously at leading order in $\omega_c\tau_{ee}$, due to our "quasihydro" limit.

To better understand the Hall viscosity, let's calculate the viscous stress tensor directly from kinetic theory...

$$\eta_{ijkl} = \langle T_{ij}|(W+W_{mag})^{-1}|T_{kl}\rangle$$

where $|\pm 2\rangle = p_F v_F\left(|T_{xx}\rangle - |T_{yy}\rangle + i\left(|T_{xy}\rangle + |T_{yx}\rangle\right)\right)$

$$\eta_{ijkl} = \begin{array}{c} \begin{pmatrix} \eta & \eta_H & \eta_H & -\eta \\ -\eta_H & \eta & \eta & \eta_H \\ -\eta_H & \eta & \eta & \eta_H \\ -\eta & -\eta_H & -\eta_H & \eta \end{pmatrix} \\ \begin{array}{cccc} xx & xy & yx & yy \end{array} \end{array}$$

$\eta_H \Rightarrow$ antisymmetric part of $\eta_{ijkl} \Rightarrow$ dissipationless!

$$\eta_{ijkl} = \eta(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl}) + \eta_H(\delta_{ik}\varepsilon_{jl} + \varepsilon_{ik}\delta_{jl})$$

We can understand how the Hall viscosity arises more abstractly. We ask what are all possible forms of the viscous stress tensor compatible with the symmetry of the isotropic Fermi liquid in the presence of a background magnetic field.

Symmetries: rotational invariance

but <u>NOT</u> parity $(x \to -x$ & $y \to y)$, as this sends $B \to -B$!

The tensor structures compatible with these symmetries are δ_{ij} & ε_{ij}.

Rotational invariance $\Rightarrow \langle T_{ij} \rangle = \langle T_{ji} \rangle \Rightarrow \eta_{ijkl} = \eta_{jikl}$.

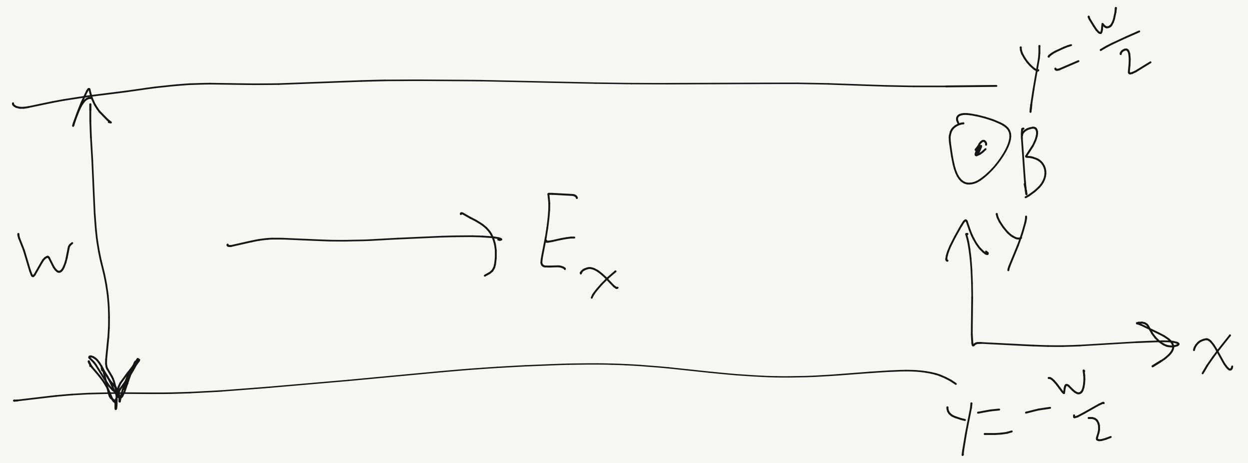
Most general possible tensor structure that can be written compatible with these symmetries is

$$\eta = \underbrace{\zeta \delta_{ij}\delta_{kl}}_{\zeta \sim \eta \left(\frac{T}{\mu}\right)^4} + \eta\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl}\right) + \eta_H \underbrace{\left(\delta_{ik}\varepsilon_{jl} + \varepsilon_{ik}\delta_{jl}\right)}_{\text{is symmetric in 2d}}$$

5.6) Hall viscosity and the Gurzhi effect

Reference: 1703.07325

Let us now look for an experimental signature of the Hall viscosity in a solid state transport experiment. We consider the same kinds of flows through narrow channels as before



The solution to the equations of motion is the same as before. If the boundary conditions are

$u_y = 0$ at $|y| = \frac{w}{2}$, and $u_x = -\zeta \frac{\partial u_x}{\partial n}$, then

$$u_y(y) = 0, \quad u_x(y) = -\frac{eE_x \tau_{imp}}{m}\left[1 - \frac{\cosh\frac{y}{\lambda}}{\cosh\frac{w}{2\lambda} + \frac{\zeta}{\lambda}\sinh\frac{w}{2\lambda}}\right]$$

The y-momentum balance equation:

$$-en\partial_y \delta_\mu + \eta_H\left(\cancelto{0}{\partial_x^2} + \partial_y^2\right)u_x = \omega_c m n u_x - \cancelto{0}{\frac{mn}{\tau_{imp}} u_y}$$

Due to the background magnetic field, we now pick up a Hall voltage!

$$\partial_y \delta\mu = -\frac{1}{en}\left[mn\omega_c u_x - \eta_H \partial_y^2 u_x\right]$$

using EOM to simplify u_x.

$$= -\frac{1}{en}\left[eBnu_x - \frac{\eta_H}{\lambda^2}\left[u_x + \frac{eE_x \tau_{imp}}{m}\right]\right]$$

$$\int_{-w/2}^{w/2} dy\, \partial_y \delta\mu = \delta\mu\left(\frac{w}{2}\right) - \delta\mu\left(-\frac{w}{2}\right) = V_H$$

Hall voltage!

$$= \frac{B}{en} I - \frac{\eta_H}{\lambda^2 (en)^2} I + \frac{E_x \tau_{imp}\eta_H}{m\lambda^2} w, \quad \text{since } I = \int_{-w/2}^{w/2} dy\,(-enu_x).$$

$$= \frac{B}{en} I - \frac{\eta_H}{\lambda^2 e^2 n^2}\left(I - \frac{E_x e^2 n \tau_{imp}}{m} w\right)$$

$$= \underbrace{\frac{B}{en} I}_{\text{ordinary Hall effect}} - \frac{\eta_H}{(\lambda en)^2} \underbrace{\left(I - \sigma_{dc} w E_x\right)}_{\text{deviation of current from Drude value!}}$$

Upon plugging in our old formula for resistivity and doing a few algebraic manipulations, we find the following result:

$$\frac{V_H}{I} = \frac{B}{en} + \frac{\eta_H}{\eta \sigma_{dc}} \frac{2\lambda \sinh\frac{w}{2\lambda}}{w\left(\cosh\frac{w}{2\lambda} + \frac{\zeta}{\lambda}\sinh\frac{w}{2\lambda} - \frac{2\lambda}{w}\sinh\frac{w}{2\lambda}\right)}$$

If $w \gg \lambda$: Hall viscosity negligible: $\frac{V_H}{I} = \frac{B}{en}$

If $w \ll \lambda$: $\frac{V_H}{I} \approx \frac{B}{en} + \frac{\eta_H}{\eta\sigma_{dc}} \frac{12\lambda^2}{w(w+6\zeta)} = \frac{B}{en} + \frac{\eta_H}{ne^2} \frac{12}{w(w+6\zeta)}$

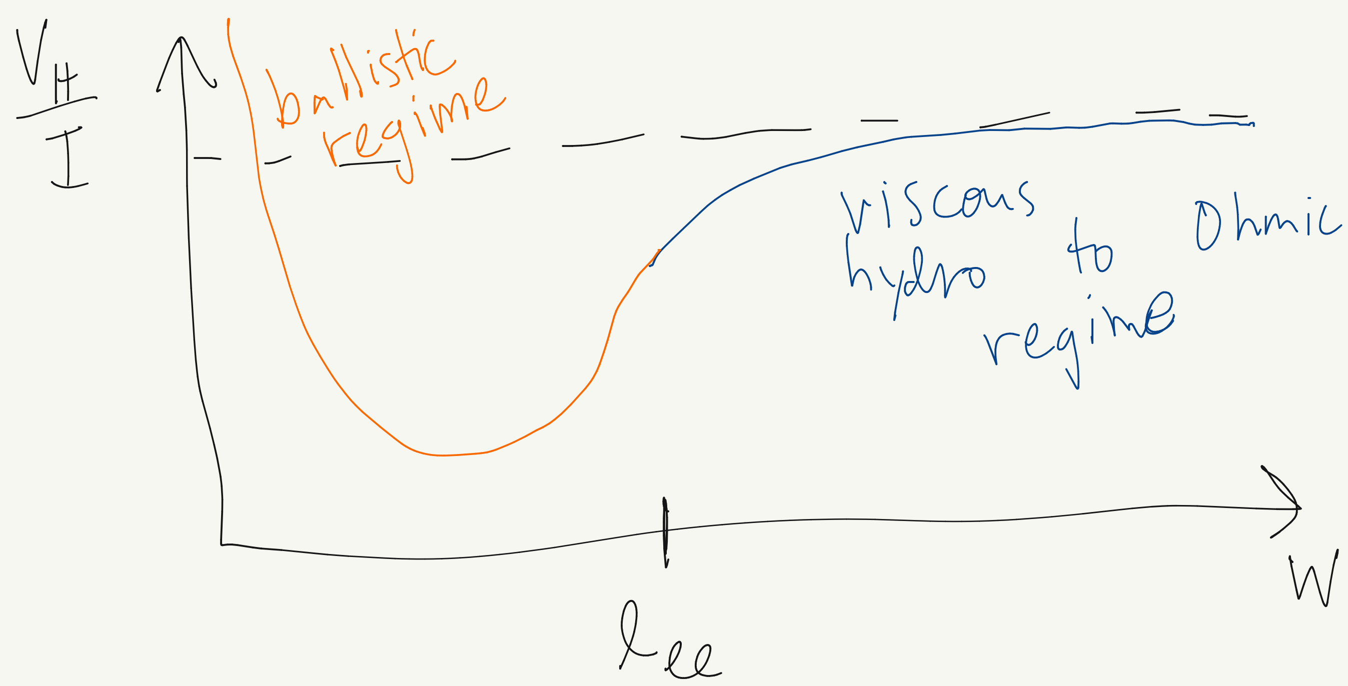
Recall that in our conventions, the Hall viscosity of the electron Fermi liquid was negative

$$\eta_H \sim -\frac{1}{2} v_F^2 \tau_{ee} \times \omega_c \tau_{ee} \qquad (\text{if } \omega_c > 0)$$

If $\rho \to 0 \ldots$

$$\frac{V_H}{I} = \frac{B}{en}\left[1 - b \underbrace{\left(\frac{v_F \tau_{ee}}{W}\right)^2}_{\text{small correction}}\right] \quad \text{in hydro regime.}$$

A numerical simulation of the full kinetic theory shows that



5.7) Viscous magnetoresistance

Reference: 1612.09275

Let's now ask what the effect of viscosity is on transport in the hydrodynamic regime in an in homogeneous metal. The kinetic calculation can be done too, but it is more complicated, so let's focus for simplicity on a viscous, isotropic low temperature Fermi liquid with negligible thermal transport (an assumption we will justify shortly)

$$\partial_i (n u_i) = 0$$

$$n\partial_i(\mu - \mu_{ext}) - \partial_j\left[\eta(\partial_i u_j + \partial_j u_i) + (\zeta - \eta)\delta_{ij}\partial_k u_k + \eta_H\left(\partial_j \varepsilon_{ik} u_k + \partial_i \varepsilon_{jk} u_k\right)\right] = -en\left(E_i + B\varepsilon_{ij} u_j\right)$$

For simplicity we will work in two dimensions

If $\mu_{ext} = 0$ and there is no inhomogeneity... the equations are solved by $\mu = 0$, $\quad u_i = -\varepsilon_{ij}\frac{E_j}{B}$.

This is precisely the classical Hall effect.

Next let's see what happens if we add a perturbatively small amount of disorder...

$$\mu_{ext}(x) \sim \delta \qquad n(x) = n_0 + \underbrace{\delta n_{imp}(x)}_{\text{small}\ldots}$$

Look for a solution to the equations of motion of the form

$$u_i = -\varepsilon_{ij}\frac{E_j}{B} + \delta\,\tilde{u}_i(x) + \delta^2 \hat{u}_i + \cdots$$

$$\mu = \mu_{ext} + \delta\,\tilde{\mu}(x) + \delta^2\hat{\mu}(x) + \cdots$$

At first order in δ:

$$0 = -\varepsilon_{ij}\frac{E_j}{B}\partial_i n_{imp} + n_0\partial_i\tilde{u}_i$$

$$n_0\partial_i\tilde{\mu} - \eta\partial_j\partial_j\tilde{u}_i - \zeta\partial_i\partial_j\tilde{u}_j - \eta_H\partial_j\partial_j\varepsilon_{ik}\tilde{u}_k = -en_0 B\varepsilon_{ij}\tilde{u}_j$$

Fourier transform: $\quad n_0 k_i\tilde{u}_i = \varepsilon_{ij}\frac{E_j}{B}k_i n_{imp}$

$$\eta k^2(ik_\ell\varepsilon_{\ell i}\tilde{u}_i) - \eta_H k^2 ik_\ell\tilde{u}_\ell = en_0 B\, ik_\ell\tilde{u}_\ell$$

$$ik_i n_0\tilde{\mu} \quad + \eta k^2\tilde{u}_i + \eta_H k^2\varepsilon_{i\ell}\tilde{u}_\ell + \zeta k_i k_j\tilde{u}_j = -en_0 B\varepsilon_{ij}\tilde{u}_j$$

Solve:

$$k_i\tilde{u}_i = \varepsilon_{ij}\frac{E_j}{B}k_i n_{imp}$$

$$k_\ell\varepsilon_{\ell i}\tilde{u}_i = \frac{1}{\eta k^2}(\eta_H k^2 + en_0 B)\varepsilon_{ij}\frac{E_j}{B}k_i n_{imp}$$

$$in_0\tilde{\mu} + (\zeta+\eta)k_i\tilde{u}_i + \eta_H\varepsilon_{i\ell}k_i\tilde{u}_\ell = \frac{-en_0 B k_i\varepsilon_{ij}\tilde{u}_j}{k^2}$$

$$\Rightarrow \quad in_0\tilde{\mu}(k) = -(\zeta+\eta)\varepsilon_{ij}\frac{E_j}{B}k_i n_{imp} - \frac{(\eta_H k^2 + en_0 B)^2}{\eta k^4}\varepsilon_{ij}\frac{E_j}{B}k_i n_{imp}$$

Now integrate momentum equation at order δ^2:

$$\int\frac{d^d x}{\mathrm{vol}}\, n_{imp}(-k)\, ik_i\tilde{\mu}(k) = \int\frac{d^d x}{\mathrm{vol}}\left(\quad B\varepsilon_{ij}\hat{J}_j\right) = \quad B\varepsilon_{ij}\hat{J}_j .$$

$$J_i^0 + \hat{J}_i = \sigma_{ij}E_j \ldots \quad \sigma_{ij} = \frac{en_0\varepsilon_{ij}}{B} + \int d^d k\,\frac{\varepsilon_{ik}k_k\varepsilon_{j\ell}k_\ell}{B^2}|n_{imp}(k)|^2\,\frac{k^4(\eta+\zeta)\eta + (\eta_H k^2 + en_0 B)^2}{\eta k^4}$$

At long wavelengths we conclude that $(k \to 0)$

$$\sigma_{xx} \approx \int d^2k \, \frac{1}{B^2} \frac{k_y^2}{\eta k^4} (e n_0 B)^2 |n_{imp}(k)|^2$$

$$\rho_{xx} \approx \frac{\sigma_{xx}}{\sigma_{xy}^2} = \int d^2k \, |n_{imp}(k)|^2 \frac{B^2 k_y^2}{\eta k^4} \sim \frac{B^2}{\eta} \sim B^2 T^2$$

This is positive magnetoresistance! This is the generic picture for transport in inhomogeneous media. In experiments in narrow channels, one can see the negative magnetoresistance arising from scattering off of the boundary.

5.8) Hydrodynamic modes in a magnetic field

The last calculation we will do in a magnetic field is to understand the fate of the hydrodynamic modes. The most interesting ones will be the sound mode and the transverse momentum mode, since the magnetic field breaks momentum conservation explicitly! So let's approximate we are in a low temperature Fermi liquid, so the bulk viscosity is rather small...

(work in $d=2$, and set electric charge $e=1$ ...

$$\partial_t \delta n + \partial_i \delta J_i = 0$$

$$m n_0 \partial_t \delta u_i + \frac{n}{\chi} \partial_i \delta n - \eta \partial_j \partial_j \delta u_i - \eta_H \partial_j \partial_j \varepsilon_{ik} \delta u_k = B \varepsilon_{ij} \delta J_j$$

$$\delta J_i = n_0 \delta u_i + \sigma_0 \left(B \varepsilon_{ij} \delta u_j - \frac{1}{\chi} \partial_i \delta n \right)$$

σ_0: incoherent conductivity

χ: charge susceptibility, $\chi = \nu$ in Fermi liquid

Looking for plane wave solutions: $e^{ikx - i\omega t}$

$$0 = \begin{pmatrix} -i\omega + k^2 \frac{\sigma_0}{\chi} & i k n_0 & i k \sigma_0 B \\ \frac{ik}{m\chi} & -i\omega + \frac{\eta}{mn} k^2 + \frac{\sigma_0 B^2}{mn} & -\omega_c + \eta_H k^2 \\ -\frac{\omega_c \sigma_0 i k}{n_0 \chi} & +\omega_c - \eta_H k^2 & -i\omega + \frac{\eta}{mn} k^2 + \frac{\sigma_0 B^2}{mn} \end{pmatrix} \begin{pmatrix} \delta n \\ \delta u_x \\ \delta u_y \end{pmatrix}$$

where $\omega_c = \frac{B}{m}$.

Let's start by looking at the limit $k \to 0$.

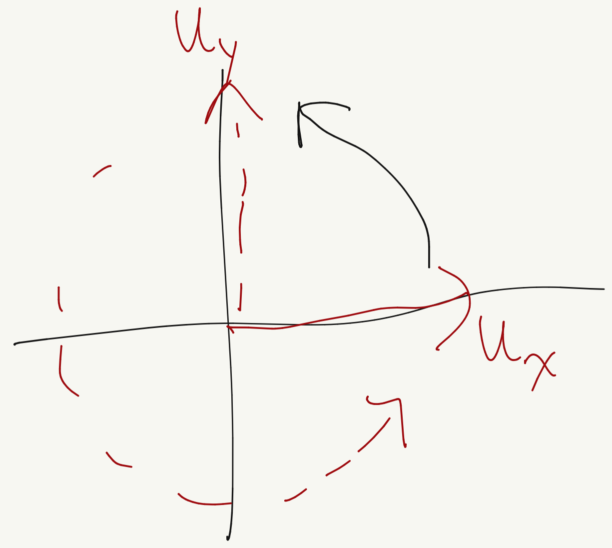
$\omega = 0$ (subleading!)

$$\det\begin{pmatrix} -i\omega & 0 & 0 \\ 0 & -i\omega+\gamma & \omega_c \\ 0 & -\omega_c & -i\omega+\gamma \end{pmatrix} = 0 \implies \omega = -i\gamma \pm \omega_c$$

damping in presence of incoherent conductivity!

cyclotron mode

The cyclotron resonance corresponds to the fact that a uniform fluid velocity will swirl around in the magnetic field due to the Lorentz force



Now let's think about that mode which was suppressed. We find that

$$\boxed{\omega = -iAk^4.}$$

has to be fluctuating since cyclotron dominated by δu_i

To calculate A, let's consider the following.

$$-i\omega\,\delta n + ik\,\delta J_x = 0$$

$$\eta k^2 \delta u_y = -B\,\delta J_x$$

$$\delta J_x = \sigma_0\left(B\,\delta u_y - \frac{1}{\chi} ik\,\delta n\right) \implies \delta u_y = \frac{ik}{B\chi}\delta n + \mathcal{O}(k^3)$$

$$\omega\,\delta n = k\,\delta J_x = -\frac{\eta k^3}{B}\delta u_y = \frac{-i\eta}{B^2\chi}k^4\,\delta n \implies \boxed{A = \frac{\eta}{B^2\chi}}$$

Hence in a magnetic field we can get subdiffusive modes!