7. Hydrodynamic correlation functions

7.1) Diffusion in linear response

Our first goal is now to argue that linear response theory in a quantum system ought to reproduce hydrodynamics in a certain regime!

Let's take the simple example of the diffusion equation for a conserved charge. Classically, we would have had

$$\partial_t \rho = D\nabla^2 \rho + \underbrace{\cdots}_{\mathcal{O}(\nabla^4)}$$

So why not expect the following in a quantum system?

$$\partial_t \langle \rho \rangle = D\nabla^2 \langle \rho \rangle + \cdots \qquad \text{w/ } \langle \rho \rangle \text{ a quantum ensemble average.}$$

Assuming this relation, what does it imply for correlation functions and linear response ?

Consider $H = H_0 - \Theta(-t) e^{\varepsilon t} \int d^d x \, \mu(x) \rho(x)$

local conserved charge

$\varepsilon > 0$ infinitesimal

external source

quantum operator

equilibrium value

$$\langle \rho(x,t) \rangle = \langle \rho \rangle + \int_{-\infty}^{0} ds \int d^d y \, \mu(y) G^R_{\rho\rho}(x-y, t-s) e^{\varepsilon s}$$

let's Fourier transform in space: (if $\vec{k} \neq \vec{0}$)

$$\langle \rho(\vec{k},t) \rangle = \int_{-\infty}^{0} ds \, \mu(\vec{k}) G^R_{\rho\rho}(\vec{k}, t-s) e^{\varepsilon s}$$

$$= \int_{-\infty}^{0} ds \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{(\varepsilon + i\omega)s - i\omega t} \mu(\vec{k}) G^R_{\rho\rho}(\vec{k}, \omega)$$

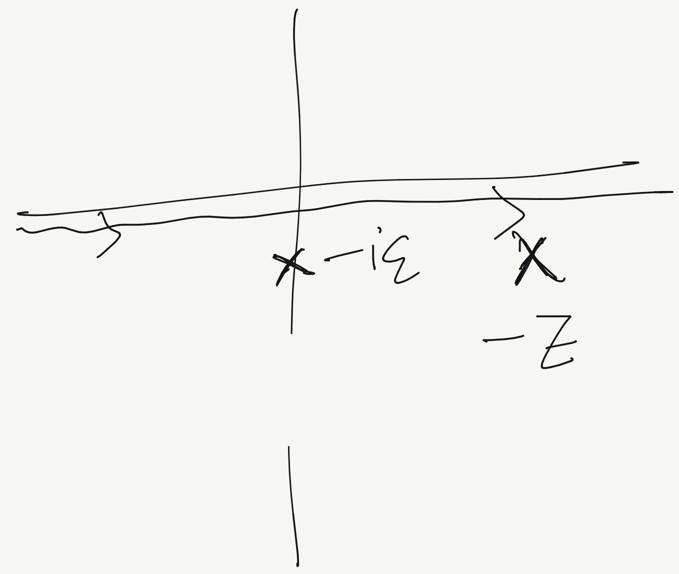
Now let's Laplace transform (since we focus on response at $t>0$):

$$\langle \rho(\vec{k},z)\rangle = \int_0^\infty dt\, e^{+izt}\langle \rho(\vec{k},t)\rangle$$

$$= \int_0^\infty dt \int_{-\infty}^{0} ds \int \frac{d\omega}{2\pi}\, e^{-izt-i\omega t+(\varepsilon+i\omega)s}\, G^R_{\rho\rho}(\vec{k},\omega)\,\mu(\vec{k})$$

$$= \int \frac{d\omega}{2\pi}\,\frac{1}{\varepsilon+i\omega}\,\frac{1}{i\omega-iz}\, G^R_{\rho\rho}(\vec{k},\omega)\,\mu(\vec{k})$$

Close integral in the lower half plane:



$$= \left[\frac{G^R_{\rho\rho}(\vec{k},0)}{-iz} + \frac{G^R_{\rho\rho}(\vec{k},z)}{iz}\right]\mu(\vec{k})$$

Now let's Laplace transform the diffusion equation:

$$\partial_t\langle \rho(\vec{k},t)\rangle = -Dk^2\langle \rho(\vec{k},t)\rangle$$

$$\int_0^\infty dt\, e^{izt}\,\partial_t\langle \rho(\vec{k},t)\rangle = -\langle \rho(\vec{k},0)\rangle - iz\langle \rho(\vec{k},t)\rangle \qquad \text{(integrate by parts)}$$

$$\langle \rho(\vec{k},0)\rangle = (Dk^2-iz)\langle \rho(\vec{k},z)\rangle = \chi_{\rho\rho}\,\mu(\vec{k}),$$

where $\chi_{\rho\rho} = G^R_{\rho\rho}(\vec{k},0) \approx$ constant as $\vec{k}\to\vec{0}$.

Combining with earlier equation....

$$\langle \rho(\vec{k},z)\rangle = \frac{\chi_{\rho\rho}\,\mu(\vec{k})}{Dk^2-iz} = \frac{G^R_{\rho\rho}(\vec{k},z)-\chi_{\rho\rho}}{iz}\,\mu(\vec{k})$$

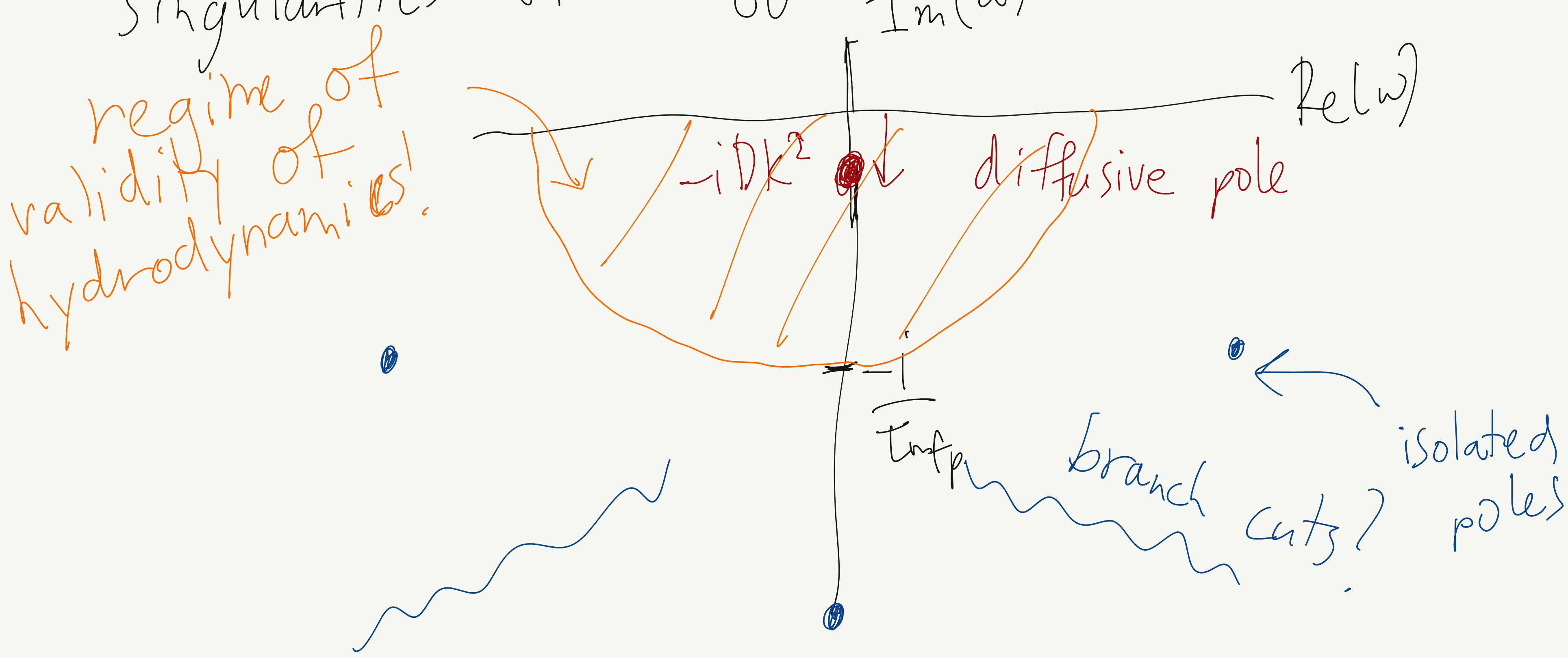
We conclude that $$\boxed{G^R_{\rho\rho}(\vec{k},\omega) \approx \chi_{\rho\rho}\, \frac{Dk^2}{Dk^2 - i\omega}}$$

What are we to make of this formula? Recall that hydrodynamics was only an approximation valid at very long length and time scales. Thus, this Green's function should only make sense at sufficiently small frequency and wave number

The highly singular nature of this Green's function has a few important features

- $G^R_{\rho\rho}(\vec{k}=\vec{0}, z) = 0$, because $\rho(\vec{k}=0) \sim Q_{tot}$ is <u>conserved</u>
- $G^R_{\rho\rho}(\vec{k}, \omega=0) = \chi_{\rho\rho}$
- The <u>diffusive pole</u> at $\omega = -iDk^2$ is the consequence of hydrodynamics in quantum correlation functions
- In general... $$G^R_{\rho\rho} = \chi_{\rho\rho} \frac{Dk^2}{(Dk^2 + D'k^4 + \dots) - i\omega} + \text{regular}(k,\omega)$$

  if $\partial_t \rho = D\nabla^2\rho - D'(\nabla^2)^2\rho + \dots$
- Let $\mathcal{O}$ be a generic operator w/ $G^R_{\mathcal{O}\rho}(\omega=0) = \chi_{\mathcal{O}\rho} \neq 0$. Then we expect the following singularities of $G^R_{\mathcal{O}\mathcal{O}}$ in the complex plane.



7.2) Kadanoff-Martin formalism

So now let us generalize this method to more generic hydrodynamic equations.

Suppose that we have the following set of hydrodynamic equations:

$$\partial_t \delta\rho_A(k) + M_{AB}(k)\delta\rho_B(k) = 0,$$ within linear response. $\{A\}$ = set of hydro modes

If we think of $M_{AB}(k)$ as Dk^2 (and assume 1 hydro mode), this is just the formalism we have gone over for diffusion. So we expect that a similar result holds.

Indeed, taking Laplace transform $\delta\rho_A(k,z) = \left[\frac{G^R_{AB}(k,z) - G^R_{AB}(k,0)}{iz}\right]\delta\mu_B(\vec{k})$

as before, and using

$$0 = -\delta\rho_A(\vec{k},0) - iz\delta\rho_A(\vec{k},z) + M_{AB}\delta\rho_B(\vec{k},z) = 0,$$ we obtain:

$$(M_{AB} - iz\delta_{AB})^{-1}\delta\rho_B(\vec{k}=0) = (M_{AB} - iz\delta_{AB})^{-1}\chi_{BC}\delta\mu_C = \frac{G^R_{AB}(\vec{k},z) - \chi_{AB}}{iz}\delta\mu_B(\vec{k})$$

$$G^R_{AB}(\vec{k},z) = \left[iz(M-iz)^{-1}\chi + \chi\right]_{AB}$$

$$= \left[(M - iz + iz)(M-iz)^{-1}\chi\right]_{AB}$$

$$\boxed{G^R_{AB}(\vec{k},z) = M_{AC}(M_{CD} - iz\delta_{CD})^{-1}\chi_{DB}}$$ (+ regular contributions...)

This formula was originally derived by Kadanoff and Martin; it is quite useful as it allows us to translate our intuition from hydrodynamics (and kinetic theory, as we will see) into quantitative predictions about the response of quantum systems.

In particular note that every pole of the Green's function will correspond to a quasinormal mode of the hydrodynamic equations!

7.3) Sound poles

The next step is to talk about sound waves in a fluid. For simplicity let's focus on only charge conservation and momentum conservation, so our equations read

$$\partial_t \delta n + \frac{1}{m}\partial_i \delta P_i - D_0 \nabla^2 \delta n = 0$$

incoherent part of current

momentum density

$$\partial_t \delta P_i + \underbrace{\partial_i \delta P}_{\text{pressure}} - \frac{1}{mn}\partial_j\left[\eta\left(\partial_j P_i + \partial_i P_j - \frac{2}{d}\delta_{ij}\partial_k P_k\right) + \zeta \delta_{ij}\partial_k P_k\right] = 0$$

Let's orient $\vec{k} = k\hat{x}$. Then we couple δn and δP_x:

$$\partial_t \delta n + \frac{ik}{m}\delta P_x + D_0 k^2 \delta n = 0$$

$$\partial_t \delta P_x + \frac{ikn}{\chi_{nn}}\delta n + \tilde{\nu} k^2 \delta P_x = 0, \quad \text{where } \tilde{\nu} = \frac{\zeta}{mn} + \left(2 - \frac{2}{d}\right)\frac{\eta}{mn}$$

So our Kadanoff-Martin matrix

$$M_{AB} = \begin{pmatrix} D_0 k^2 & ik/m \\ \frac{ikn}{\chi_{nn}} & \tilde{\nu} k^2 \end{pmatrix}, \qquad \chi_{AB} = \begin{pmatrix} \chi_{nn} & 0 \\ 0 & mn \end{pmatrix}$$

since we defined this to be momentum susceptibility

Let's calculate

$$G^R_{nn}(\vec{k}, \omega) = M_{nA}(M_{AB} - i\omega\delta_{AB})^{-1}\chi_{nn}$$

$$(M_{AB} - i\omega\delta_{AB})^{-1} = \begin{pmatrix} D_0 k^2 - i\omega & ik/m \\ \frac{ikn}{\chi_{nn}} & \tilde{\nu} k^2 - i\omega \end{pmatrix}^{-1}$$

$$= \frac{1}{k^2 \frac{n}{m\chi_{nn}} + (D_0 k^2 - i\omega)(\tilde{\nu} k^2 - i\omega)} \begin{pmatrix} \tilde{\nu} k^2 - i\omega & -\frac{ik}{m} \\ -\frac{ikn}{\chi_{nn}} & D_0 k^2 - i\omega \end{pmatrix}$$

$$G^R_{nn}(\vec{k},\omega) = \chi_{nn} \frac{D_0 k^2(\tilde{\nu}k^2 - i\omega) + k^2 v_s^2}{k^2 v_s^2 + (\tilde{\nu}k^2 - i\omega)(D_0 k^2 - i\omega)} + \cdots, \text{ where } v_s^2 = \frac{n}{m\chi_{nn}}$$

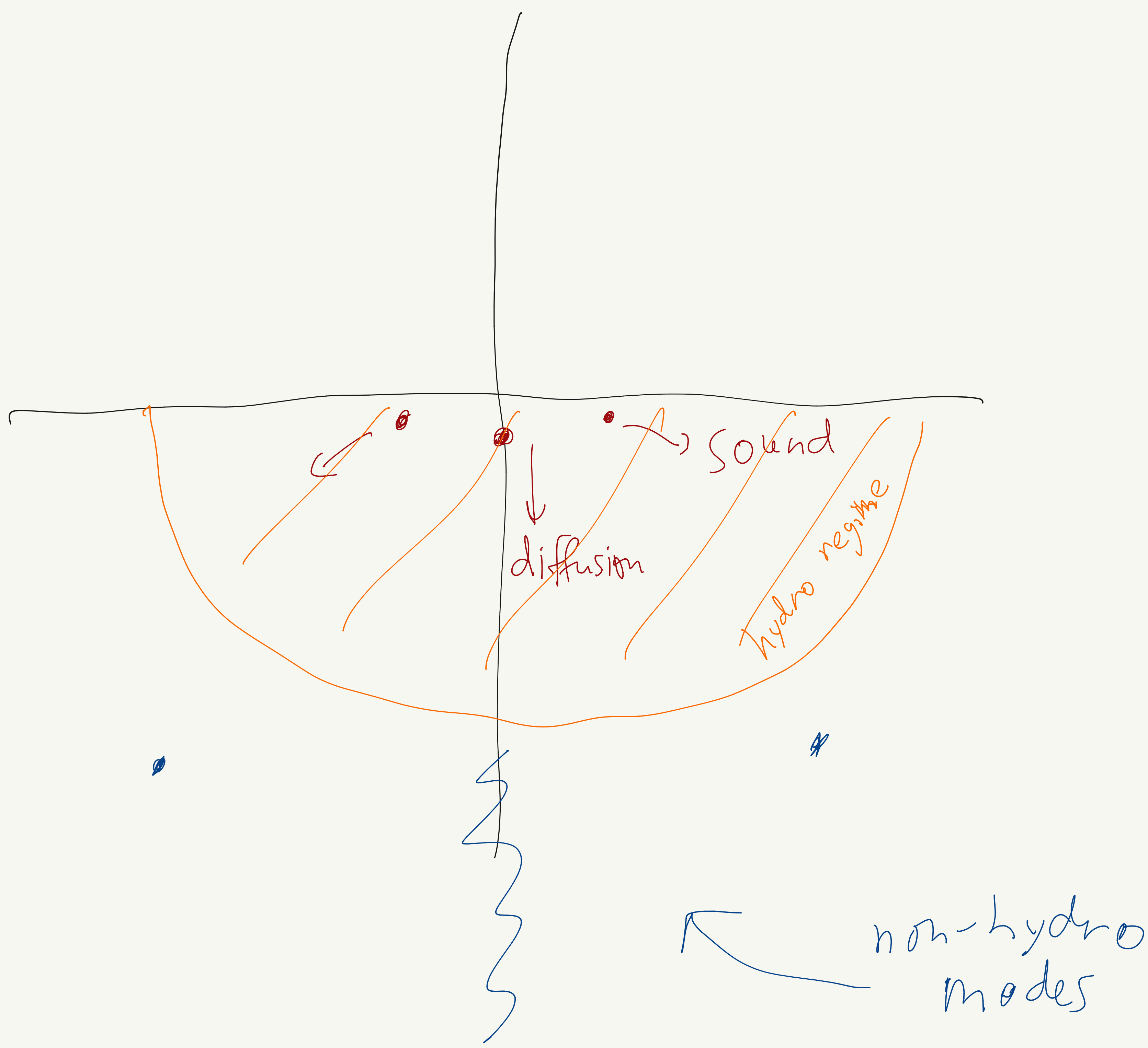
As before we can make a few generic comments.

- $G^R_{nn} \sim k^2$ as $k \to 0$; since $[n(\vec{k}=0), H] = 0$ this had to happen as a consequence of conservation laws.
- As in the diffusion case, the poles of G^R_{nn} occur at the frequencies where classical sound waves exist

Transverse components of momentum are diffusive, and thus analogously to before:

$$G^R_{P_y P_y}(\vec{k},\omega) = \frac{\eta k^2}{\frac{\eta}{mn} k^2 - i\omega}$$

Putting this together we can sketch the motion of poles in the complex plane:



7.4) Quasihydrodynamics and the Drude peak

We can also follow apply the Kadanoff-Martin formalism to systems with almost (but not quite) conserved quantities. For example let's think about a fluid with weak momentum relaxation

For the normal fluid... $\vec{k} = k\hat{x}$

$$\chi_{AB} = \begin{pmatrix} \chi_{nn} & 0 & 0 \\ 0 & mn & 0 \\ 0 & 0 & mn \end{pmatrix} \qquad M_{AB} = \begin{pmatrix} D_0 k^2 & \frac{ik}{m} & 0 \\ \frac{ikn}{\chi_{nn}} & \frac{1}{\tau}+\tilde{\nu}k^2 & 0 \\ 0 & 0 & \frac{1}{\tau}+\nu k^2 \end{pmatrix}$$

where $\tilde{\nu} = \left(\frac{2d-2}{d}\eta + \zeta\right)\frac{1}{mn}$, $\nu = \frac{\eta}{mn}$.

The sound modes are contained in the charge correlator...

$$G^R_{nn}(\vec{k},\omega) = \left[1 + i\omega(M - i\omega)^{-1}\right]_{nA}\chi_{An}$$

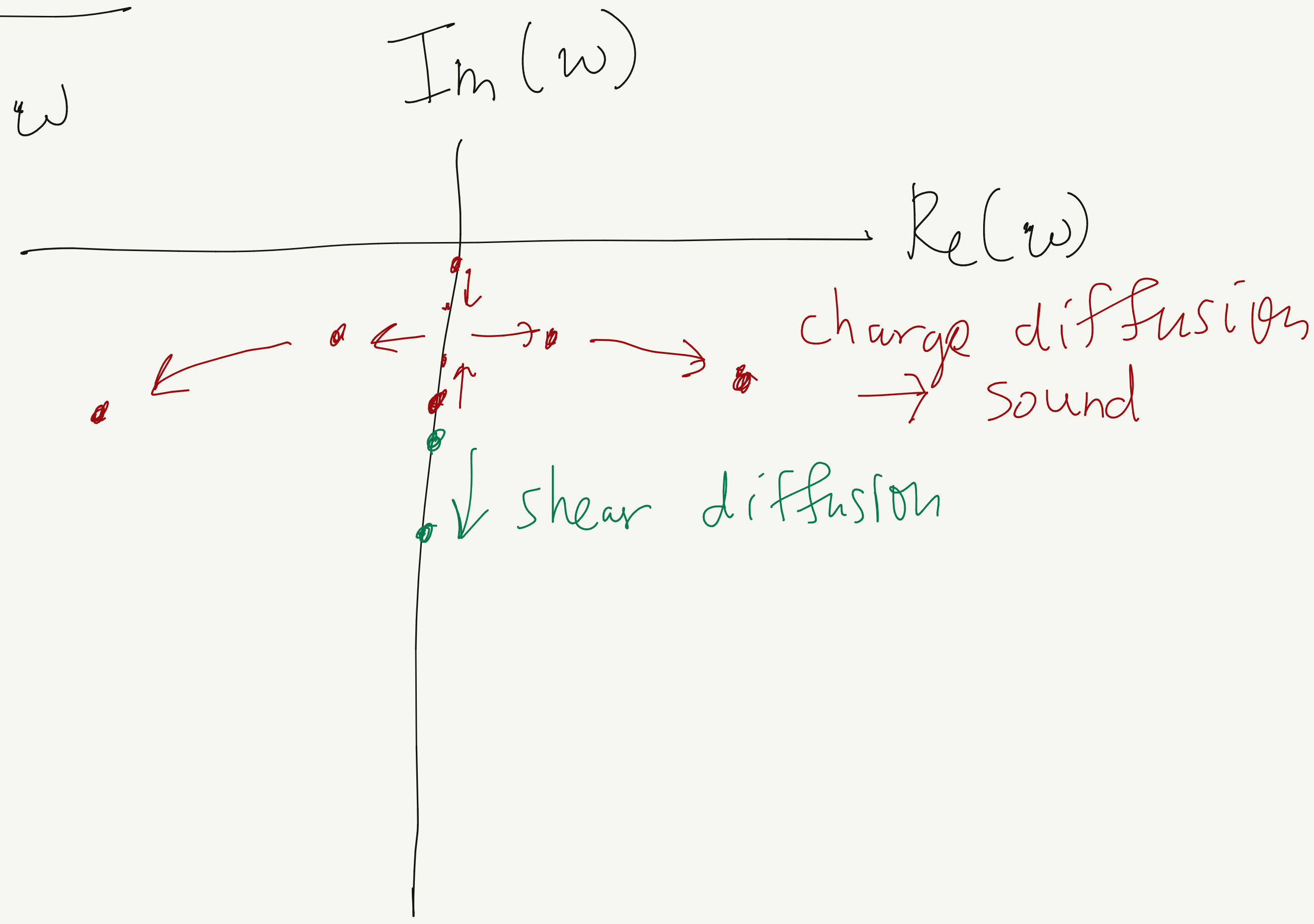
$$= \chi_{nn}\left(1 + i\omega\frac{\frac{1}{\tau}+\tilde{\nu}k^2 - i\omega}{(D_0k^2 - i\omega)(\frac{1}{\tau} - i\omega + \tilde{\nu}k^2) + k^2v_s^2}\right)$$

$$= \chi_{nn}\frac{k^2\left[v_s^2 + D_0\left(\frac{1}{\tau} + \tilde{\nu}k^2 - i\omega\right)\right]}{(D_0k^2 - i\omega)\left(\frac{1}{\tau} - i\omega + \tilde{\nu}k^2\right) + k^2v_s^2}$$

The transverse momentum correlator is given by

$$G^R_{p_yp_y}(k,\omega) = mn\frac{\frac{1}{\tau}+\nu k^2}{\frac{1}{\tau}+\nu k^2 - i\omega}$$

The motion of poles in the complex plane...



As a simple example, let's now compute the charge conductivity

$$\sigma_{yy}(\omega) = \frac{1}{i\omega}\left[G^R_{J_yJ_y}(i\omega) - G^R_{J_yJ_y}(0) \right].$$

since $\chi_{J_xP_x} = -en$

$\chi_{P_xP_x} = mn$

(think kinetic theory)

In a general fluid, we expect that $J_x = \frac{-e}{m} P_x + \underbrace{J_x^{inc}}_{\text{not a hydro mode}} \ldots$

$$\text{since } G^R_{J_yJ_y} = \left(\frac{e}{m}\right)^2 G^R_{P_yP_y} - \underbrace{\frac{e}{m}\left(G^R_{J_y^{inc}P_y} + G^R_{P_yJ_y^{inc}}\right) + G^R_{J_y^{inc}J_y^{inc}}}_{\text{non-hydrodynamic sectors, will be regular...}}$$

$$\sigma_{yy}(\omega) = \frac{1}{i\omega}\left(\frac{e}{m}\right)^2 \left[G^R_{P_yP_y}(i\omega) - G^R_{P_yP_y}(0) \right] + \text{regular terms}$$

$$= \frac{1}{i\omega}\left(\frac{e}{m}\right)^2 mn \left[\frac{\frac{1}{\tau}}{\frac{1}{\tau} - i\omega} - \frac{1/\tau}{1/\tau} \right]$$

$$= \frac{ne^2}{m} \frac{1}{\frac{1}{\tau} - i\omega}$$

This is the so called Drude peak. It is commonly used as a model for the AC response of a metal. It is quantitatively accurate only in the limit of weak momentum relaxation. A quantitatively accurate Drude model in a metal is possible, yet is surprisingly hard to find in the real world due to various inter band transitions, the relative importance of incoherent effects, and so on.

It's also worth noting that if $\tau \to \infty$ ...

$$\sigma_{yy}(\omega) \to \frac{ne^2}{m}\left[\frac{1}{-i\omega} + \pi\delta(\omega) \right]$$

coefficient of this δ-function is thermodynamic

$$\frac{ne^2}{m} = \frac{\chi^2_{J_xP_x}}{\chi_{P_xP_x}}$$

7.5) Kinetic theory as quasihydrodynamics

Finally let's think of the Boltzmann equation in a "quasihydrodynamic limit", so that we can calculate Green's functions of systems with long lived quasiparticles.

Kinetic equations:

collision integral

$$\partial_t f_p + \vec{v}\cdot\nabla_x f_p = -W[f]$$

← perturbatively small when collisions are weak.

Can we write this in Kadanoff-Martin form?

Let's first calculate susceptibility. Since $\rho_{\text{non-interacting}} = e^{-\beta[\sum \varepsilon_p n_p]}$

— ε_p "conjugate" to n_p!

$$\chi_{pq} = \frac{\partial n_p}{\partial \varepsilon_q} = \delta_{pq}\left(\frac{\partial n_p}{\partial \varepsilon_p}\right) = \delta_{pq}\left(-\frac{\partial f_F}{\partial \varepsilon}\right)_p$$

$$\sim \langle p|q\rangle !$$

$$(\partial_t + \vec{v}\cdot\nabla_x)\chi_{pq}\Phi_q = -W_{pq}\Phi_q \quad \text{where } \langle p|W|q\rangle = W_{pq}.$$

So Kadanoff-Martin's: $M_{pq} = \vec{v}\cdot\nabla_x\delta_{pq} + (\chi^{-1}W)_{pq}$

Let's go to our kinetic theory work-horse model: 2d FL at low T.

Work in angular harmonic basis where $\langle m|\chi|m'\rangle = \nu\,\delta_{mm'}$

← density of states

$$\langle m|\vec{v}\cdot i\vec{k}|m'\rangle = ik\frac{v_F}{2}\left(\delta_{m,m'+1} + \delta_{m,m'-1}\right)$$

$$\langle m|W|m'\rangle = \frac{\nu}{\tau_{ee}} \times \begin{cases} 1 & |m|\geq 2 \\ 0 & |m|\leq 1 \end{cases}.$$

As an example, let's compute Green's functions of the operators

$$n = \sum n_p, \qquad J_y = -e v_F \sum \sin\theta\, n_p$$

$$\to |0\rangle \qquad\qquad \to -\frac{e v_F}{2i}\left(|1\rangle - |-1\rangle\right)$$

Now recall that

$$(W + ik\cdot v - i\omega)^{-1} = \begin{pmatrix} -i\omega & ikv_F/\sqrt{2} & 0 \\ \frac{ikv_F}{\sqrt{2}} & Y(k,\omega) - i\omega & 0 \\ 0 & 0 & Y(k,\omega) - i\omega \end{pmatrix}$$

$$|0\rangle \qquad |x\rangle = \frac{|1\rangle + |-1\rangle}{\sqrt{2}} \qquad |Y\rangle \quad \frac{|1\rangle - |-1\rangle}{\sqrt{2}\, i}$$

where $Y(k,\omega) = \frac{1}{2}\left(-\left(\frac{1}{\tau_{ee}} - i\omega\right) + \sqrt{\left(\frac{1}{\tau_{ee}} - i\omega\right)^2 + (kv_F)^2}\right)$

So $$G^R_{p_y p_y} = \frac{e^2 v_F^2 \nu}{2} \times \left[1 + i\omega \times \frac{1}{Y(k,\omega) - i\omega}\right]$$

$$= \frac{e^2 v_F^2 \nu}{2} \frac{Y(k,\omega)}{Y(k,\omega) - i\omega}$$

$$= \frac{e^2 v_F^2 \nu}{2} \times \frac{Dk_x^2}{Dk_x^2 - \frac{i}{2}\left(1 - i\omega\tau_{ee} + \sqrt{(1 - i\omega\tau_{ee})^2 + (kv_F\tau_{ee})^2}\right)}$$

where $D = \frac{1}{4} v_F^2 \tau_{ee}$.

Note the presence of the branch cuts in the Green's function — this is a generic effect in kinetic theory

$$G^R_{nn} = \nu \times \left(1 + i\omega \frac{Y - i\omega}{(Y - i\omega)(-i\omega) + k^2 v_s^2}\right), \text{ where } v_s^2 = \frac{v_F^2}{2}$$

$$= \frac{\nu k^2 v_s^2}{k^2 v_s^2 - \omega^2 - \frac{i\omega}{2}\left[\sqrt{k^2 v_F^2 + \left(\frac{1}{\tau_{ee}} - i\omega\right)^2} - \left(\frac{1}{\tau_{ee}} - i\omega\right)\right]}$$