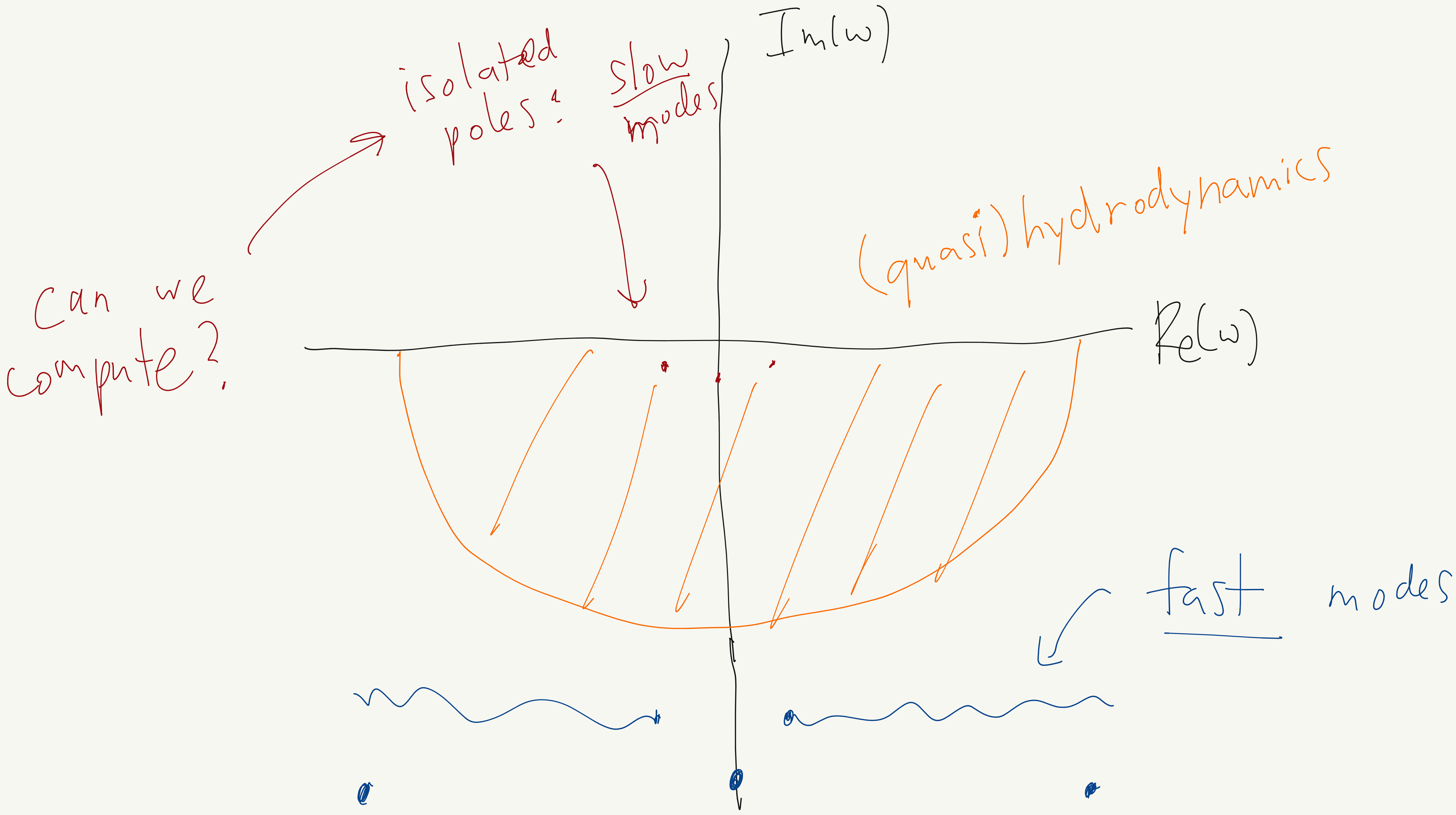
8. Memory matrix formalism

8.1) Slow and fast modes

Our goal is now to introduce a method for isolating the (quasi)hydrodynamic poles found in Green's functions up until this point. Let us quickly review the problem of interest



This means that we expect that for a generic operator $\mathcal{O}$ that "overlaps" w/ one of these (quasi)hydro slow modes...

$$G^R_{\mathcal{O}\mathcal{O}} = \sum \frac{\#}{\omega - \omega_{\text{slow}}} + \underbrace{\text{regular [near } \omega \to 0]}_{\text{contains fast modes}}$$

ω_{slow} ← location of pole

- What are the coefficients #?
- If ω_{slow} is well separated from other time scales, how to compute it?

8.2) An inner product

Our goal is to answer these questions, but before this we need to introduce a bit of technology to help us calculate things. Let's start with the notion of operator overlap, which we'd like to make precise.

Recall in our discussion of kinetic theory... we had

$$\langle \mathcal{O}_1 | \mathcal{O}_2 \rangle = \int \frac{d^d p}{(2\pi\hbar)^d} \left(-\frac{\partial f_{eq}}{\partial \varepsilon}\right)_p \mathcal{O}_1(p)\,\mathcal{O}_2(p) \quad ; \quad \text{where } \mathcal{O}_{1,2} = \int d^d p\, \mathcal{O}_{1,2}(p)\, n_p$$

$$= \int \frac{d^d p}{(2\pi\hbar)^d}\, d^d p'\, \chi_{n_p n_{p'}}\, \mathcal{O}_1(p)\,\mathcal{O}_2(p') = \chi_{\mathcal{O}_1 \mathcal{O}_2}$$

We also saw that this inner product was extremely useful – this inner product, giving "overlap" between 2 operators, helped to control which operators were sensitive to which collision rates: e.g. if

$\langle J_x | P_x \rangle \neq 0$ and $W = \Gamma |P_x\rangle\langle P_x| + \gamma(1 - |P_x\rangle\langle P_x|)$, $\Gamma \ll \gamma$, then

$$\sigma_{xx} = \langle J_x | W^{-1} | J_x \rangle \sim \frac{\langle J_x | P_x \rangle^2}{\Gamma \langle P_x | P_x \rangle} + \cdots$$

$\uparrow$ — dominates transport.

Is there a quantum analogue of this inner product?

Claim: Define

$$(A|B) = \int_0^\beta \frac{d\lambda}{\beta} \langle A\, B(i\lambda) \rangle = \int_0^\beta \frac{d\lambda}{\beta} \frac{\mathrm{tr}\left(e^{-(\beta-\lambda)H} A\, e^{-\lambda H} B\right)}{Z(\beta)}$$

Then $T\chi_{AB} = (A|B)$.

Proof: We begin by taking a time derivative:

$$C_{AB}(t) = (A(t)|B)_\beta = \int_0^\beta \frac{d\lambda}{\beta} \frac{\mathrm{tr}\left(e^{-(\beta-\lambda-it)H} A e^{-(\lambda+it)H} B\right)}{Z(\beta)}$$

$$\partial_t C_{AB}(t) = \int_0 \frac{d\lambda}{\beta Z}\, i\, \underbrace{\mathrm{tr}\left(e^{-(\beta-\lambda-it)H}[H,A]e^{-(\lambda+it)H}B\right)}_{= \frac{\partial}{\partial\lambda}\mathrm{tr}\left(e^{-(\beta-\lambda-it)}Ae^{-(\lambda+it)H}B\right)}$$

$$\partial_t C_{AB}(t) = \frac{i}{\beta Z}\mathrm{tr}\left(A(t)e^{-\beta H}B - e^{-\beta H}A(t)B\right)$$

$$\Theta(t)\partial_t C_{AB}(t) = -\frac{1}{\beta} G^R_{AB}(t)\,!$$

So we see that G^R is deeply related to $C_{AB}(t)$. Moreover,

$$\chi_{AB} = G^R_{AB}(\omega=0) = \int_{-\infty}^{\infty} dt\, G^R_{AB}(t) = -\frac{1}{T}\int_{-\infty}^{\infty} dt\, \Theta(t)\partial_t C_{AB}(t)$$

$$= -\frac{1}{T}\int_0^{\infty} dt\, \partial_t C_{AB}(t) = \frac{1}{T} C_{AB}(t=0)!$$

It's also instructive to take the Fourier transform of G^R_{AB}:

$$-T G^R_{AB}(z) = \int_{-\infty}^{\infty} dt\, e^{izt}\partial_t C_{AB}(t) = -C_{AB}(t=0) - iz\, C_{AB}(z)$$

$$\Rightarrow \quad C_{AB}(z) = \frac{T}{iz}\left(G^R_{AB}(z) - G^R_{AB}(0)\right).$$

The Laplace transform of C relates to $G^R(z)$!

Recall that the electrical conductivity was

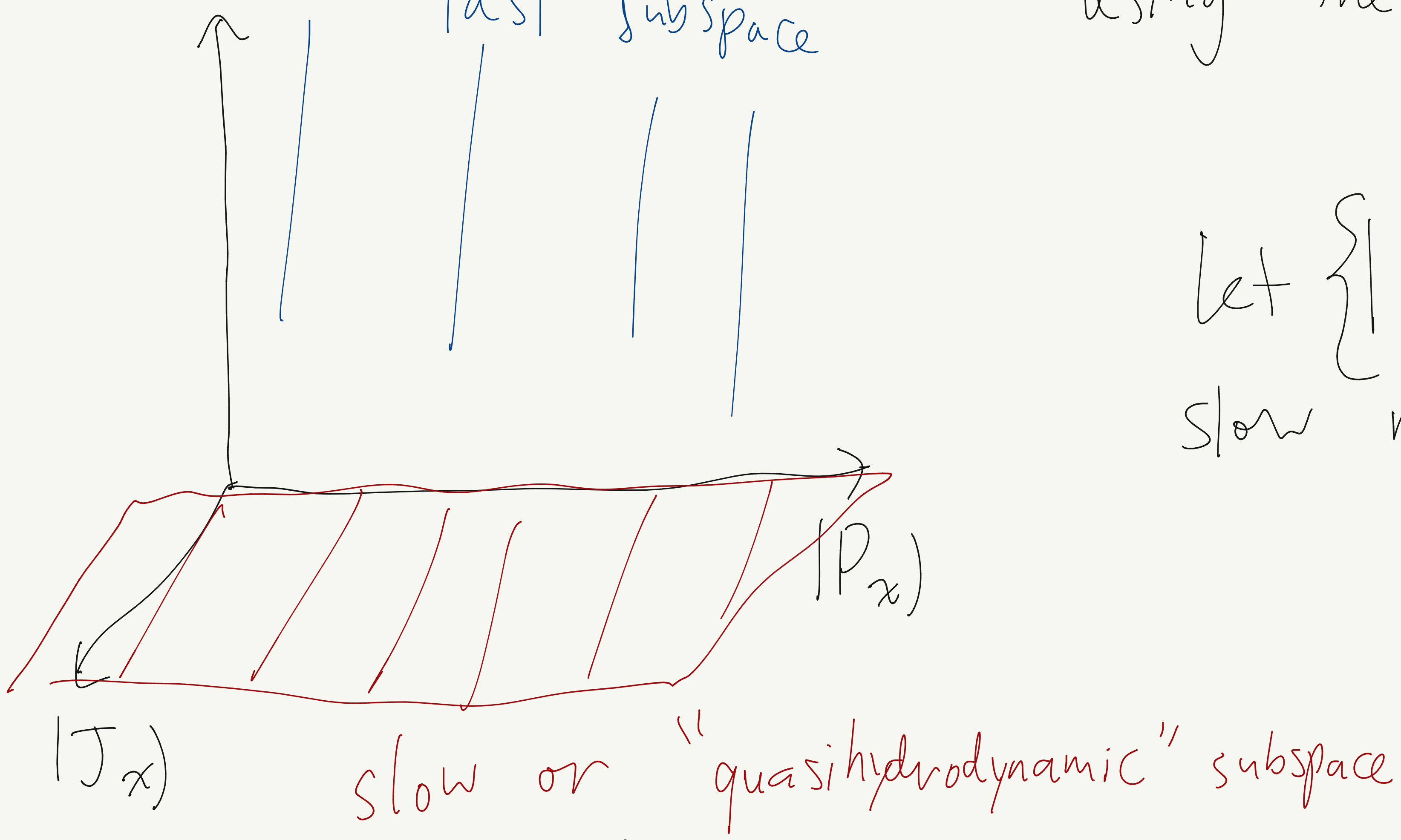
$$\sigma_{xx}(\omega) = \frac{1}{i\omega}\left(G^R_{J_xJ_x}(i\omega) - G^R_{J_xJ_x}(0)\right) = \frac{1}{T}C_{J_xJ_x}(\omega)!$$

So in addition to giving us our natural inner product on quantum operators, with the overlap itself related to the thermodynamic susceptibility, we have also found a natural way of calculating the conductivity!

8.3) The memory matrix

With this formula for the conductivity in mind, we are now ready to calculate. The first thing we need to do is formalize the notion of fast and slow operators. This will be similar to how we derived hydrodynamics from the linearized kinetic theory, but in a fully quantum mechanical language

Step 1: Separate out quantum operators into "fast" and "slow", using the inner product $(A|B)$...



Let $\{|A)\}$ denote the set of slow modes.

We can write formal projectors onto slow (p) and fast ($q = 1-p$):

$$p = \sum_{\substack{A,B\\ \text{slow}}} |A)\left((A|B)^{-1}\right)(B| = \frac{1}{T}\sum_{\substack{A,B\\ \text{slow}}} |A)\chi^{-1}_{AB}(B|$$

Since by definition, $|\mathcal{O})$ is fast if $(B|\mathcal{O}) = 0$ for any slow $|B)$, and if $|C)$ is slow...

$$p|C) = \frac{1}{T}\sum_{A,B} |A)\chi^{-1}_{AB}\underbrace{(B|C)}_{=T\chi_{BC}} = \sum_{A,B} |A)\delta_{AC} = |C),$$

p is a good projector!

Step 2: We want to calculate $C_{AB}(z)$. It is useful to write this as follows. Define the Liouvillian $\mathcal{L}$ by

$$\partial_t |A(t)) = i\mathcal{L}|A(t)) : \quad \text{namely,} \quad \mathcal{L}|A) = |[H,A]).$$

Note that $\mathcal{L}$ is antisymmetric:

$$(A|\mathcal{L}|B) = \int_0^\beta \frac{d\lambda}{\beta Z} \mathrm{tr}\left(e^{-(\beta-\lambda)H} A e^{-\lambda H}(HB - BH)\right)$$

using cyclic trace properties

$$= \int_0^\beta \frac{d\lambda}{\beta Z} \mathrm{tr}\left(e^{-(\beta-\lambda)H}(AH - HA)\, e^{-\lambda H} B\right) = -(B|\mathcal{L}|A)\,!$$

Then we can write, using $(A(t)| = (A|e^{-i\mathcal{L}t}$:

$$\int_0^\infty dt\, e^{izt} C_{AB}(t) = \int_0^\infty dt\, (A|e^{i(z-\mathcal{L})t}|B) = (A|i(z-\mathcal{L})^{-1}|B)\,!$$

Step 3: Our next goal is to evaluate $\sigma_{xx} = (J_x|i(z-\mathcal{L})^{-1}|J_x)$ as a block matrix inverse. Suppose that we DEMAND that $|J_x)$ is a slow degree of freedom. Then we may write

$$z - \mathcal{L} = \begin{array}{cc} \begin{pmatrix} z - p\mathcal{L}p & -p\mathcal{L}q \\ -q\mathcal{L}p & z - q\mathcal{L}q \end{pmatrix} & \begin{array}{c} p \\ q \end{array} \\ \begin{array}{cc} \quad p \quad & \quad q \quad \end{array} & \end{array}$$

Using block matrix identities from before: (and $\mathcal{L}^T = -\mathcal{L}$)

$$p(z-\mathcal{L})^{-1}p = p\left(z - p\mathcal{L}p + p\mathcal{L}^T q(z - q\mathcal{L}q)^{-1} q\mathcal{L}p\right)^{-1} p$$

Now let's define some matrices on only the slow degrees of freedom.

As before: $$\chi_{AB} = \frac{1}{T}(A|B)$$

New things: $$N_{AB} = \frac{i}{T}(A|\mathcal{L}|B) = -N_{BA}$$

$$M_{AB} = \frac{-i}{T}(A|p\mathcal{L}\, q(z-q\mathcal{L}q)^{-1} q\mathcal{L}p|B)$$

memory matrix!

Since $$\sigma_{AB}(\omega) = \frac{1}{T}\sum_{\substack{C,D \\ \text{slow}}} (A|C)\left(-iz + ip\mathcal{L}p - ip\mathcal{L}^{T} q(z-q\mathcal{L}q)^{-1} q\mathcal{L}p\right)^{-1}_{CD}(D|B)$$

$$\boxed{\sigma_{AB}(\omega) = \chi_{AC}\left(-iz\chi + N + M\right)^{-1}_{CD}\chi_{DB}}$$

This is our main result for this final part of the course. It tells us how to integrate out "fast" degrees of freedom to calculate Green's functions of slow degrees of freedom. Remarkably, the identities used to get this formula are exact — there isnt a need for slow modes to actually be slow! But as we will see shortly, this method does work best when we actually do use slow modes as slow...

8.4) The memory matrix in real time

It can be instructive to re derive this formula directly in real time, to get more physical intuition for what we have actually done here (and for the name memory matrix). First, we prove a lemma

Lemma: $$e^{(A+B)t} = e^{At} + \int_0^t ds\, e^{As} B e^{(A+B)(t-s)} \qquad (A, B \text{ matrices})$$

Proof: $$\partial_t e^{(A+B)t} = e^{(A+B)t}(A+B)$$

$$\partial_t(\text{RHS}) = e^{At}A + e^{At}B + \int_0^t ds\, e^{As} B e^{(A+B)(t-s)}(A+B) = (\text{RHS})(A+B)$$

At $t=0$, both matrices are the identity. They obey same ODE w/ same initial condition, therefore are the same!

Now let's use this identity writing $i\mathcal{L} = \underbrace{i\mathcal{L}_q}_{A} + \underbrace{i\mathcal{L}_p}_{B}$

A, B slow!

$$C_{AB}(t) = (A|\left[e^{i\mathcal{L}_q t} + \int_0^t ds\, e^{i\mathcal{L}_q(t-s)}\, i\mathcal{L}_p\, e^{i\mathcal{L}s}\right]|B)$$

$$= i\int_0^t ds\, (A|e^{i\mathcal{L}_q(t-s)}\underbrace{\mathcal{L}_p e^{i\mathcal{L}s}\, |B)}_{=\sum_{\text{slow } C,D} \frac{1}{T}|C)\chi^{-1}_{CD} C_{DB}(s)\ !}$$

$$\partial_t C_{AB}(t) = i(A|\mathcal{L}_p e^{i\mathcal{L}t}|B) - \int_0^t ds\, \underbrace{(A|p\mathcal{L}q\, e^{iq\mathcal{L}q(t-s)}\, q\mathcal{L}p|B(s))}_{\text{added projectors for presentation purposes...}}$$

"N" term, "rotation" of slow modes amongst themselves

"M" term: integrating out fast modes requires keeping track of their feedback!

The origin of the name "memory matrix" is thus because the memory matrix represents the time dependent feedback of the fast modes on the slow modes, after we try to integrate them out. In general, due to the linearity of quantum mechanics, we can always solve for the isolated dynamics of a reduced subset of the degrees of freedom without sacrificing anything! But the price we pay is non locality in time of the resulting equations. The memory matrix as we will see, is useful when the modes we have integrated out (and the resulting nonlocality...) occur on fast time scales relative to those of interest. Then an approximate locality in time can re-emerge, but with dissipation now in the picture!

8.5) Transport with weak momentum relaxation
Reference: 1612.07324

Let's now turn to an explicit example of how we can use the memory matrix to calculate interesting things about transport. We return to our canonical example of a quantum system with a continuous translation invariance that is weakly broken by some external source

$$H = \underbrace{H_0}_{\text{translation invariant}} - \int d^d x \underbrace{h(x)}_{\text{perturbatively small parameter!}} \mathcal{O}(x)$$

Here translation invariance means that there exists a momentum operator P_x such that $[P_x, H_0] = 0$, and $\underbrace{[P_x, \mathcal{O}] = -i\partial_x \mathcal{O}}_{P_x \text{ generates spatial translations, as } H \text{ generates time translations}}$

Let's evaluate the electrical conductivity of such a system using the memory matrix. For simplicity, assume rotational invariance. We let the slow modes be J_x and P_x.

Proposition: $(J_x | P_x) = \rho$, the total charge density.

Heuristic Proof: Note that charge conservation holds as an operator identity:

$$\frac{\partial \rho}{\partial t} + \partial_i J_i = 0.$$

Assume operators vary only in x & t. Now consider

$$\chi_{\underbrace{J_x(x)}_{\text{local}} \underbrace{P_x}_{\text{global}}} = \int_{-\infty}^{\infty} dt\, G^R_{J_x(x) P_x}(t) = \int_0^{\infty} dt\; i\langle [J_x(t), P_x] \rangle$$

$$= \int_0^{\infty} dt \langle -\partial_x J_x(t) \rangle = \int_0^{\infty} dt \left\langle \frac{\partial \rho}{\partial t} \right\rangle = \langle \rho \rangle ! \quad \blacksquare$$

Note: If we integrate over space: $\int dx\, \chi_{J_x P_x} = \mathrm{Vol} \times \langle \rho \rangle$, and we conventionally divide out by such overall volume factors: i.e.

$$\chi_{J_x(k=0)\, P_x(k=0)} = \langle \rho \rangle \quad \text{if} \quad P_x(k=0) = \frac{1}{\sqrt{\mathrm{Vol}}} \int dx\, P_x(x), \text{ etc.}$$

Hence, "slow" modes J_x & P_x are not independent, and we'll need to be slightly careful. Next, let's calculate M & N.

$N = 0$ for us, by <u>time reversal</u> <u>symmetry</u>.

The calculation of M is more interesting.

$$M_{P_x P_x} = (\dot{P}_x | q\, i(z - q\mathcal{L}q)^{-1} q | \dot{P}_x) \qquad (\text{since } q\mathcal{L}p|P_x) = -iq|\dot{P}_x))$$

$$\dot{P}_x = i[H, P_x] = \underbrace{i\cancel{[H_0, P_x]}}_{\text{by translation inv.}}^{\;0} - i\int d^d x\, \underbrace{h(x)}_{\text{external source, not dynamical!}} [\underbrace{\mathcal{O}(x)}_{\text{operator}}, P_x]$$

$$= \int d^d x\, h(x)\, \partial_x \mathcal{O}(x) = -\int d^d x\, \mathcal{O}(x)\, \partial_x h(x)$$

Since h was perturbatively small parameter, we conclude that $M_{P_x P_x} \sim h^2$!

More explicitly:

$$M_{P_x P_x}(z) = \int \frac{d^d x\, d^d x'}{\mathrm{Vol}} (\partial_x h)(x) (\partial_x h)(x') \underbrace{(\mathcal{O}(x) | q\; i(z - q\mathcal{L}q)^{-1} q | \mathcal{O}(x'))}$$

$$= \underbrace{(\mathcal{O}(x) | i(z - \mathcal{L})^{-1} | \mathcal{O}(x'))}_{\text{use translation invariant Hamiltonian at leading order in } h! \text{ because}} + \mathcal{O}(h)$$

$$(\mathcal{O}(x) | q(z - \mathcal{L}q)^{-1} q | \mathcal{O}(x')) = (\mathcal{O}(x) | (z - \mathcal{L} + \mathcal{L}p)^{-1} | \mathcal{O}(x'))$$

$$= (\mathcal{O}(x) | (z - \mathcal{L})^{-1} [1 - \mathcal{L}p(z - \mathcal{L}q)^{-1}] | \mathcal{O}(x')), \text{ and } \underbrace{p}_{k=0 \text{ modes}} (z - \mathcal{L}q)^{-1} | \underbrace{\mathcal{O}(x')}_{k \neq 0 \text{ mode}}) = \mathcal{O}(h)$$

$$M_{P_xP_x}(z) = \int \frac{d^dk}{(2\pi)^d}\, k_x^2\, |h(\vec{k})|^2\, \underbrace{(\mathcal{O}(k)|\, i(z-\mathcal{L})^{-1}\, |\mathcal{O}(\vec{k}))}_{=\frac{1}{iz}\left[G^R_{\mathcal{OO}}(\vec{k},z) - G^R_{\mathcal{OO}}(\vec{k},0)\right]}$$

(using time reversal) $= \lim_{\omega\to 0} \frac{1}{\omega}\, \mathrm{Im}\left(G^R_{\mathcal{OO}}(\vec{k},\omega)\right)$, as $z\to 0$

So the momentum memory matrix is given by the spectral weight of the operator which broke translation invariance! This is a very useful fact, which as we will soon see, allows for tractable computations in a number of theories...

$$M_{J_xP_x}(z) = (\dot{J}_x|\, q\, i(z-q\mathcal{L}q)^{-1} \int d^dx\, (-\partial_x h(x))\, |\mathcal{O}(x))$$

$k=0$ mode, at leading order

$k\neq 0$ modes only!

Translation invariance of H_0 implies that, like before, we need an extra $h\,\mathcal{O}$ in $(z-q\mathcal{L}q)^{-1}$ to make $k=0$ & $k\neq 0$ modes overlap. Thus $M_{J_xP_x} = \mathcal{O}(h^2)$.

In general, since J_x not conserved, we have $M_{J_xJ_x} = \mathcal{O}(h^0)$.

Now let's put all of this together. Using the scaling for M that we have found, we have

$$\sigma_{J_xJ_x}(\omega) = \chi_{J_xA}\underbrace{(M_{AB} - i\omega\chi_{AB})^{-1}}\chi_{BJ_x}$$

$$= \left[\begin{array}{cc|c} \multicolumn{2}{c}{\begin{pmatrix} h^2a & h^2b \\ h^2b & h^0c \end{pmatrix}} & \begin{array}{c} P_x \\ J_x \end{array} \\ P_x & J_x & \end{array}\right]^{-1} \sim \begin{pmatrix} \frac{1}{h^2a} & -\frac{b}{ac} \\ -\frac{b}{ac} & \frac{1}{c} \end{pmatrix}$$

this mode dominates!

$$\boxed{\sigma_{J_xJ_x}(\omega) = \frac{\chi^2_{J_xP_x}}{M_{P_xP_x} - i\omega\chi_{P_xP_x}}}$$

We have essentially given a quantum mechanical, rigorous perturbative derivation of the Drude formula! If we define

$$\chi_{P_xP_x} = m n, \quad \chi_{J_xP_x} = \rho = -en$$

effective mass; number density of electrons

and $$\frac{1}{\tau_{imp}} = \frac{1}{mn} M_{P_xP_x} \quad \ldots \quad \boxed{\sigma_{xx}(\omega) = \frac{ne^2\tau}{m} \times \frac{1}{1-i\omega\tau}}$$

Our formula for the momentum relaxation time is a non-quasiparticles generalization of Fermi's golden rule for impurity scattering — here it is the spectral weight of the inhomogeneously sourced operator that sets the disorder scattering rate.

8.6) Spectral weights and momentum relaxation times

Let's put this new formula to use, and start making predictions for the resistivity of various systems by estimating the spectral weight that will control the momentum relaxation time.

Example 1: Hydrodynamics. Let's consider a low T Fermi liquid, where we previously calculated

$$G^R_{\rho\rho} = \frac{\chi_{\rho\rho} k^2 v_s^2}{k^2 v_s^2 - \omega^2 - i\omega \tilde{\nu} k^2}$$

$$\tilde{\nu} \sim \frac{\eta}{mn}\left(2 - \frac{2}{d}\right)$$

If we couple a system to an inhomogeneous potential...

$$\rho_{xx} \sim \frac{m}{ne^2\tau}, \quad \text{where } \frac{1}{\tau} \sim \int d^dk \; k_x^2 |\mu(k)|^2 \times \lim_{\omega\to 0} \frac{\operatorname{Im} G^R_{\rho\rho}(k,\omega)}{\omega}$$

$$\sim \int d^dk \; k_x^2 |\mu(k)|^2 \times \frac{\chi_{\rho\rho} k^2 v_s^2 \tilde{\nu} k^2}{(k^2 v_s^2)^2 + 0}$$

$$\sim \int d^dk \; k_x^2 |\mu(k)|^2 \eta$$

which is precisely the inhomogeneous Gurzhi effect from before.

We can also see that diffusive poles dominate the spectral weight as $k \to 0$... if

$$G^R_{\rho\rho}(\vec{k},\omega) = \frac{ADk^2}{Dk^2 - i\omega}, \quad \lim_{\omega\to 0} \frac{\mathrm{Im}(G^R_{\rho\rho}(\vec{k},\omega))}{\omega} \sim \frac{A}{Dk^2}!$$

So, ~~for~~ example, thermal diffusion in the FL dominates as $k \to 0$, albeit the coefficient $\frac{A}{D} \sim T^2$ is suppressed...

Example 2: Quantum critical (scale invariant!) models.

Suppose we have a system described by a scale invariant QFT where the dimensional analysis takes the following form:

$$t \to \lambda^z t \qquad (z = \text{dynamic critical exponent})$$

$$x \to \lambda x \qquad (\text{e.g. } \omega = k^z \text{ excitations?})$$

$$\mathcal{O} \to \lambda^{-\Delta}\mathcal{O} \qquad \Delta = \text{scaling dimension of } \mathcal{O}!$$

so $G^R_{\mathcal{O}\mathcal{O}}(\vec{k},\omega) \sim \int d^dx \, dt \, e^{ik\cdot(x-y)-\omega t)} \langle [\mathcal{O}(x,t), \mathcal{O}\ ] \rangle$

$\Big\downarrow$ dimensional analysis

$$G^R_{\mathcal{O}\mathcal{O}} \sim G^R_{\mathcal{O}\mathcal{O}} \times \lambda^{d+z-2\Delta}$$

Since $T \to \lambda^{-z}T$ under rescaling as $T \sim \text{energy} \sim \frac{1}{\text{time}}$...

we estimate that

$$\lim_{\omega\to 0} \frac{1}{\omega}\mathrm{Im}(G^R_{\mathcal{O}\mathcal{O}}) \sim \frac{1}{T}T^{(2\Delta - z - d)/z} \sim T^{(2\Delta - 2z - d)/z}$$

Now, let's imagine we have Gaussian disorder, so $|\mu(k)|^2 \sim$ const?

Then $$\frac{1}{\tau} \sim \underbrace{\int d^d k\, k^2}_{\sim T^{\frac{d+2}{z}}} \times \lim_{\omega\to 0} \frac{G^R_{OO}}{\omega} \sim \boxed{T^{2(1+\Delta-z)/z}}$$

This gives resistivity of critical theories perturbed by disorder!

In models of non-Fermi liquids with a Fermi surface, the exponents can be more subtle. See 1408.6549, 1401.7012

8.7) Magnetic fields

Next, let us (quickly) justify the form of the conductivity tensor in the presence of a magnetic field. For simplicity we stick to two spatial dimensions.

Since magnetic field breaks time reversal, we need to write

$\sigma_{AB}(\omega) = \chi_{AC}(M+N-i\omega\chi)^{-1}_{CD}\chi_{DB}$, where $N_{AB} = \frac{i}{T}(A|\mathcal{L}|B)$

Let's begin by thinking about a theory w/ no disorder & momentum conservation [when $B=0$]. We have

$$H = H_0 - \int d^2x\, A_i J_i, \quad \text{w/} \quad A_i = \frac{B}{2}\varepsilon_{ij}x_j \ \text{(e.g.)}$$

$\dot{P}_i = i[H, P_i] = B\varepsilon_{ij}J_j$, which can be proved by (e.g.)

$$\dot{P}_x = i[H, P_x] = -i\int d^2x\,(-Bx)[J_y, P_x] = -B\int d^2x\; x\,\partial_x J_y = BJ_y$$

Hence $N_{P_iP_j} = \chi_{P_i\dot{P}_j} = B\varepsilon_{jk}\chi_{P_iJ_k} = -B\rho\,\varepsilon_{ij}$

Similarly, $N_{J_iP_j} = -B\chi_{JJ}\varepsilon_{ij}$

When $\omega = 0$, we have

$$\sigma_{ij}(\omega) = \chi_{J_i A} (M+N)^{-1}_{AB}\, \chi_{B J_j}$$

$$M+N = \begin{array}{c} \left(\begin{array}{cc|cc} M_J & -N_J & 0 & -B\chi_{JJ} \\ N_J & M_J & B\chi_{JJ} & 0 \\ \hline 0 & -B\chi_{JJ} & 0 & -B\rho \\ B\chi_{JJ} & 0 & B\rho & 0 \end{array}\right) \\ \begin{array}{cccc} J_x & J_y & P_x & P_y \end{array} \end{array}$$

unimportant residual matrix

$$= \left(\begin{array}{c|c} \tilde{M} & 0 \\ \hline 0 & 0 \end{array}\right) + \chi \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \underbrace{\begin{matrix} 0 & -\frac{B}{\rho} \\ \frac{B}{\rho} & 0 \end{matrix}}_{Z} \end{array}\right) \chi$$

$$\left(\begin{array}{c|c} \sigma_{ij} & - \\ \hline - & - \end{array}\right) = \left(\begin{array}{c|c} a^2 \tilde{M} & ab\tilde{M} \\ \hline ab\tilde{M} & Z + b^2\tilde{M} \end{array}\right)^{-1}, \text{ where } \quad \begin{aligned} a &= (\chi^{-1})_{JJ} \\ b &= (\chi^{-1})_{JP} \end{aligned}$$

$$\sigma_{ij} = \left(Z + b^2\tilde{M} - \frac{(ab)^2}{a^2}\tilde{M}\right)^{-1} = Z^{-1}$$

$$\sigma_{ij} = \frac{\rho}{B}\varepsilon_{ij}$$

This is precisely the prediction we expect from the classical/quantum Hall effect.

If we now try to go back and include the momentum relaxing scattering processes, we find that

If $B \sim h^2 \sim 1/\tau$ is perturbatively small, we get that

$$\sigma_{ij} = \rho^2 \begin{pmatrix} M_{P_xP_x} & -B\rho \\ B\rho & M_{P_xP_x} \end{pmatrix}^{-1},$$ in agreement w/ our Drude prediction

If the magnetic field is not perturbatively small, then we find that

$$(\sigma^{-1})_{xx} \approx \frac{M_{P_x P_x}}{\rho^2},$$ where the spectral weight can be evaluated at finite B

This result requires a lot of algebra so we won't show explicitly.

8.8) Hydrodynamics and the memory matrix

Now we will use the memory matrix to provide a more rigorous derivation of the form of hydrodynamic Green's functions, focusing for simplicity on the diffusive Green's function

If we only have a single conservation law, e.g. energy diffusion, let's define our single slow mode to be

$$\varepsilon(\vec{k}) = \frac{1}{\sqrt{\mathrm{Vol}}}\int d^d x \; e^{-ik\cdot x}\varepsilon(x), \text{ if } H = \int d^d x \; \varepsilon(x).$$

We know that $\partial_t \varepsilon(\vec{k}) + i\vec{k}\cdot\vec{J}_E(\vec{k}) = 0$

energy current operator is NOT conserved!

Now by the memory matrix formalism,

$$C_{\varepsilon(\vec{k})\varepsilon(\vec{k})}(z) = C(z) = \frac{\chi^2_{\varepsilon(k)\varepsilon(k)}}{M_{\varepsilon(k)\varepsilon(k)}(z) - iz\chi_{\varepsilon(k)\varepsilon(k)}}$$

[no N by time reversal!]

In general, $\chi_{\varepsilon(k)\varepsilon(k)} = \chi$ is well-behaved as $x \to 0$.
After all AT $k=0$ ...

$$\chi = \frac{1}{\mathrm{Vol}}\frac{\mathrm{tr}\left(e^{-\beta H}(H-\langle H\rangle)^2\right)}{Z(\beta)} = \frac{\langle H^2\rangle - \langle H\rangle^2}{\mathrm{Vol}} = \frac{1}{\mathrm{Vol}}\left(-\frac{\partial^2}{\partial\beta^2}\log Z(\beta)\right)$$

thermal energy density

$$= -\frac{\partial\langle\varepsilon\rangle}{\partial\beta} = T^2\frac{\partial\varepsilon}{\partial T} = T^2 c$$

specific heat

Now let's think about the form of the memory matrix

$$M(z) = (\dot{\varepsilon}(k)|\, q\, i(z - q\mathcal{L}q)^{-1} q\, |\dot{\varepsilon}(k))$$

must be a "fast" quantity... this vector can't overlap w/ $|\varepsilon(k))$

$$= k_i k_j (J_E^i(k)|\, i(z - q\mathcal{L}q)^{-1} |J_E^j(k))$$

$$= k_i k_j \Sigma_{ij}(z)$$

We now claim that Σ_{ij} is finite as $z \to 0$. Indeed this is sensible, since J_E is not a conserved degree of freedom and there are no hydrodynamic poles. Note if rotational invariant... $\Sigma_{ij}(z) = \Sigma(z)\delta_{ij}$

Hence we conclude $$C(z) = \frac{\chi^2}{-iz\chi + k^2\Sigma} = \frac{1}{iz}\Big(G^R(z) - \underbrace{G^R(0)}_{\chi}\Big)$$

$$\Rightarrow G^R_{\varepsilon\varepsilon}(\vec{k}, z) = \frac{k^2\Sigma\chi}{k^2\Sigma - iz\chi} = \frac{\chi D k^2}{Dk^2 - iz} \quad \text{if } D = \frac{\Sigma}{\chi}$$

Hence we precisely recover the diffusive form of the hydrodynamic Green's function that we predicted previously.

Lastly, let's observe that the thermal conductivity is given by

$$T\bar{\kappa} = \lim_{\omega \to 0} \frac{1}{i\omega}\left[G^R_{J_E^x J_E^x}(\omega) - G^R_{J_E^x J_E^x}(0)\right] = \frac{1}{T} C_{J_E^x J_E^x}(0)$$

$$= \frac{1}{T}\Sigma_{ij}$$

The thermal diffusion constant is $D = \frac{\Sigma}{\chi} = \frac{T^2\bar{\kappa}}{T^2 c} = \frac{\bar{\kappa}}{c}$

which is the canonical Einstein relation

Similar derivations work for more complicated hydrodynamic theories with multiple diffusion modes, or sound modes, etc. one can also include the almost conserved quantities in this language to provide a formal justification of quasihydrodynamics

8.9) Derivation of the linearized Boltzmann equation

Our last goal for this course is to use the memory matrix to sketch a semi rigorous derivation of the quantum Boltzmann equation in linear response

$$H = \underbrace{\sum_p \varepsilon_p c_p^\dagger c_p}_{H_0,\ \text{unperturbed problem}} + \underbrace{\sum U_{p_1p_2p_3p_4} c_{p_1}^\dagger c_{p_2}^\dagger c_{p_3} c_{p_4}}_{\text{perturbatively weak interactions.}}$$

Our slow modes correspond to the quadratic modes $c_p^\dagger c_q$. let's organize these slow modes as

$c_{p+k/2}^\dagger c_{p-k/2} = n_p(k)$ (definition of $n_p(k)$). Using $\{c_p^\dagger, c_q\} = \delta_{pq}$,

$$i[H_0, n_p(\vec{k})] = i\left(\varepsilon_{p+k/2} c_{p+k/2}^\dagger c_{p-k/2} - \varepsilon_{p-k/2} c_{p+k/2}^\dagger c_{p-k/2}\right)$$
$$\simeq (i\vec{k}\cdot\vec{v})\, n_p(\vec{k})!$$

So the N matrix in the memory matrix formalism:

$$N_{n_p(k)n_q(k)} = \frac{i}{T}(n_p(k)|\mathcal{L}|n_q(\vec{k})) = \frac{1}{T} i\vec{k}\cdot\vec{v}_{\vec{p}}\,(n_p(k)|n_q(k)) = i\vec{k}\cdot\vec{v}_p \chi_{\vec{p}}\, \delta_{\vec{p}\vec{q}}$$

The memory matrix M comes from considering

$$M_{n_pn_q}(z) = \frac{i}{T}(\dot{n}_p(k)|q(z - q\mathcal{L}q)^{-1}q|\dot{n}_q(k)).$$ First note that

$$q\,\dot{n}_q(k) = i[U_{p_1p_2p_3p_4} c_{p_1}^\dagger c_{p_2}^\dagger c_{p_3} c_{p_4}, n_q(k)] \sim \sum U c^\dagger c^\dagger c c$$

project out 2-fermion operators!

(won't keep track of indices in this sketch)

Just like in our treatment of weak momentum relaxation, we write

$\mathcal{L}_0 = [H_0, \cdot]$ and

$$M_{n_p n_q}(z) \approx \underbrace{\frac{i}{T}(n_p(k)|q(z-\mathcal{L}_0)^{-1} q|n_q(k))}_{\text{This object is } \lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}\left(\underbrace{G^R_{c^\dagger c^\dagger cc,\, c^\dagger c^\dagger cc}}_{\text{neglecting sum over indices...}}(\omega)\right)|U|^2} + \mathcal{O}(\overset{\text{weak interactions}}{U^3})$$

That spectral weight can be evaluated by a tedious "field theory" calculation. It gives

$$\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}\left(G^R_{c_1^\dagger c_2^\dagger c_3 c_4,\, c_1^\dagger c_2^\dagger c_3 c_4}(\omega)\right) = f_F(\varepsilon_1) f_F(\varepsilon_2)(1-f_F(\varepsilon_3))(1-f_F(\varepsilon_4)) \times \pi\beta\,\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)$$

which was precisely part of... the coefficient that showed up in our collision integral $\langle p|W|q\rangle$!

In the Kadanoff-Martin formalism, we found that

$$G^R_{n_p n_q}(\vec{k},\omega) = \chi_{n_p n_r}\left(1 + i\omega\left(W_{n_r n_{r'}} + i\vec{k}\cdot\vec{v}_{\vec{r}}\,\chi_{\vec{r}\vec{r}'} - i\omega\chi_{\vec{r}\vec{r}'}\right)^{-1}\chi_{\vec{r}\vec{r}'}\right),$$

implying that

$$C_{n_p n_q}(\vec{k},\omega) = \frac{1}{i\omega}\left(G^R_{n_p n_q}(\omega) - \underbrace{G^R_{n_p n_q}(0)}_{\chi_{\vec{p}\vec{q}}}\right)$$

$$= \left[\chi\left(W + i\vec{k}\cdot\vec{v}\chi - i\omega\chi\right)^{-1}\chi\right]_{n_{\vec{p}} n_{\vec{q}}}!$$

This precisely agrees with our prior identification of the streaming terms in the kinetic equations as N, and the linearized collision integral as M