Homework 1

Due: 3:00 PM, Friday, September 20.

Problem 1 (Baber scattering): Consider a conductor with an electron band and a hole band. The number density of electrons/holes in equilibrium is $n_{\rm e,h}$, and the effective masses of electrons and holes are $m_{\rm e,h}$ respectively. In general, $m_{\rm e} \neq m_{\rm h}$. Let $\mathcal{P}_{\rm e,h}$ denote the momentum density of the electrons and holes, respectively. The current $J_{\rm e,h}$ of electrons/holes is given by

$$J_{\rm e} = -\frac{e}{m_{\rm e}}\mathcal{P}_{\rm e}, \tag{1a}$$

$$J_{\rm h} = \frac{e}{m_{\rm h}}\mathcal{P}_{\rm h}. \tag{1b}$$

Suppose that there are three scattering mechanisms in the problem: electron-hole ($\gamma_{\rm e,h}$), electron-impurity ($\Gamma_{\rm e}$), and hole-impurity ($\Gamma_{\rm h}$). Assume the canonical temperature dependences for electron-electron (and electron-hole!) and electron-impurity scattering. Finally, assume that while impurity collisions are momentum relaxing, electron-hole collisions conserve the combined momentum of electrons and holes.

(a) Following the logic that we used to derive the Drude model, argue that Newton's laws take the form

$$en_{\rm e}E = -\Gamma_{\rm e}\mathcal{P}_{\rm e} - \gamma_{\rm e}\mathcal{P}_{\rm e} + \gamma_{\rm h}\mathcal{P}_{\rm h}, \tag{2a}$$

$$-en_{\rm h}E = -\Gamma_{\rm h}\mathcal{P}_{\rm h} - \gamma_{\rm h}\mathcal{P}_{\rm h} + \gamma_{\rm e}\mathcal{P}_{\rm e}. \tag{2b}$$

(Since the Drude model (and this one) appear (at this time) highly phenomenological, you should really be taking these equations as *defining* $\gamma_{\rm e,h}$ and $\Gamma_{\rm e,h}$. So your primary purpose here is to justify the crude model above, given the physical assumptions described above.)

From now on, you can take $\Gamma_{\rm e} = \Gamma_{\rm h} = \Gamma$, $\gamma_{\rm e} = \gamma_{\rm h} = \gamma$, and $\Gamma \ll \gamma$. This limit can be realistic in low density semiconductors. Also assume that $n_{\rm e,h}$ and $m_{\rm e,h}$ are temperature independent.

(b) Calculate the electrical resistivity ρ. Argue that ρ will be approximately temperature independent unless $n_{\rm e} \approx n_{\rm h}$, in which case $\rho \propto T^2$. This mechanism for $\rho \propto T^2$ described in this problem is called **Baber scattering**. What is the physical origin of Baber scattering?

(c) If the material has two electron bands, is it possible to find an analogue of Baber scattering, with $\rho \propto T^2$ even if $\Gamma \ll \gamma$? Explain your answer intuitively, and check it by generalizing the above model.

Problem 2 (Planckian bounds): A rather popular idea in the literature as of late is that a metal with strongly correlated electrons has a Planckian limited resistivity, where the Drude transport time obeys the bound

$$\tau_{\rm tr} \gtrsim \frac{\hbar}{k_{\rm B}T}. \tag{3}$$

Like the Mott-Ioffe-Regel bound, the inequality here is only "heuristic"; the bound is not rigorous. The motivation for this bound is either (*i*) that it "follows" from the Heisenberg energy-time uncertainty principle, or (*ii*) that it follows from dimensional analysis, assuming that all microscopic scales like the Fermi energy are irrelevant for low energy dynamics and transport.

(a) Many metals, even at room temperature, can appear to come close to saturating (3). Based on the temperature dependence, what would be the "canonical" explanation for the resistivity?

(b) Is (3) always obeyed in a metal, in the real world?

(c) Above what temperature T would it be possible for a metal to violate the Mott-Ioffe-Regel bound but obey (3)? Estimate the answer in Kelvin for metals with $v_{\rm F} = 10^4, 10^5, 10^6$ m/s. Is it possible to observe this effect experimentally?

Problem 3 (Improving the relaxation time approximation): Here we derive another bound on the resistivity in our formal kinetic theory of transport.

(a) Explain why the fluctuations in the number density of fermionic excitations is given by $\langle \Phi | \mathsf{n} \rangle$, where

$$|\mathsf{n}\rangle = \int \mathrm{d}^d\mathbf{p}\, |\mathbf{p}\rangle. \tag{4}$$

(b) Explain why if carrier number is a conserved quantity, then

$$\langle \mathbf{p}|\mathsf{W}|\mathbf{p}\rangle = \frac{1}{(2\pi\hbar)^d}\left(-\frac{\partial f_{\rm F}(\mathbf{p})}{\partial\epsilon}\right)\left[g(\mathbf{p},\mathbf{p}') - \delta(\mathbf{p}-\mathbf{p}')\int \mathrm{d}^d\mathbf{q}\, g(\mathbf{p},\mathbf{q})\right] \tag{5}$$

where g is a "symmetric" function:

$$\left(-\frac{\partial f_{\rm F}(\mathbf{p})}{\partial\epsilon}\right) g(\mathbf{p},\mathbf{q}) = \left(-\frac{\partial f_{\rm F}(\mathbf{q})}{\partial\epsilon}\right) g(\mathbf{q},\mathbf{p}) \tag{6}$$

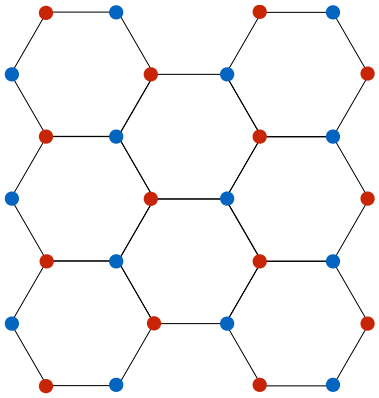
(c) Consider the modified (linearized) collision integral

$$\langle \mathbf{p}|\widetilde{\mathsf{W}}|\mathbf{p}'\rangle = -\delta(\mathbf{p}-\mathbf{p}')\frac{1}{(2\pi\hbar)^d}\left(-\frac{\partial f_{\rm F}(\mathbf{p})}{\partial\epsilon}\right)\int \mathrm{d}^d\mathbf{q}\, g(\mathbf{p},\mathbf{q}). \tag{7}$$

Let ρ be the resistivity with collision integral W and $\widetilde{\rho}$ be the resistivity with $\widetilde{\mathsf{W}}$. Find a simple expression for $\widetilde{\rho}$, and interpret the answer as a generalized relaxation time approximation. For simplicity, you can assume that the resistivity is isotropic.

(d) Show that $2\widetilde{\rho} \geq \rho$.

Problem 4 (Thermoelectric transport in graphene): Graphene is a two dimensional honeycomb lattice of carbon:



A tight-binding model for conduction electrons is given by

$$H = -t \sum_{i \sim j} c_{i\sigma}^\dagger c_{j\sigma}, \tag{8}$$

where $t \approx 2.8$ eV. The lattice spacing between carbon atoms is $a \approx 0.14$ nm. The unit cell of the honeycomb lattice consists of one red and one blue atom, as shown in the figure.

(a) First find the Brillouin zone for the honeycomb lattice. Use that the vectors separating adjacent points in the lattice are

$$\mathbf{a}_1 = a\hat{\mathbf{x}}, \tag{9a}$$

$$\mathbf{a}_2 = a\frac{\sqrt{3}\hat{\mathbf{y}} - \hat{\mathbf{x}}}{2}, \tag{9b}$$

$$\mathbf{a}_3 = a\frac{-\sqrt{3}\hat{\mathbf{y}} - \hat{\mathbf{x}}}{2}. \tag{9c}$$

(b) Show that the dispersion relation is

$$\epsilon_\sigma(\mathbf{k}) = \pm t \left| \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{a}_1} + \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{a}_2} + \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{a}_3} \right|. \tag{10}$$

(c) Based on where carbon lies in the periodic table, argue that it is natural for "charge neutral" graphene to have exactly one electron per carbon atom in the conduction band described by (8). What is the Fermi energy? Explain why, at low temperatures, it is acceptable to approximate the dispersion of graphene with

$$\epsilon_a(\mathbf{p}) = \pm \hbar v_\mathrm{F} |\mathbf{p}| \tag{11}$$

where $a = 1, \ldots, N$ and $\mathbf{p}$ a (relative) momentum. What is N? Some different "flavors" of fermion may come from spin, while others may come from the band structure. What is v_F, including its (estimated) numerical value? At what temperature does (11) break down?

(d) Using the qualitative form of the band structure (11), together with the relaxation time approximation, use the Boltzmann equation to calculate σ_{ij}, α_{ij} and $\bar{\kappa}_{ij}$ as a function of chemical potential μ and temperature T, along with the parameters v_F and relaxation time τ.

(e) Plot $\mathcal{L}/\mathcal{L}_0$ as a function of μ/T. Explain why the Wiedemann-Franz and Mott laws are violated when $|\mu| \lesssim T$. (This effect is called **bipolar diffusion**.) Can κ/σ deviate arbitrarily from the Wiedemann-Franz law?

When electron-electron interactions in graphene become strong, it can be more sensible to use a "hydrodynamic" description for thermoelectric transport. Due to the "relativistic" nature of the dispersion relation, the momentum density and the energy current are essentially identical, up to a factor of Fermi velocity:

$$J_\mathrm{E} = \mathcal{P} v_\mathrm{F}^2. \tag{12}$$

We can follow the spirit of the Drude model to come up with an alternative theory of transport, where the charge current is

$$J = \frac{-en}{W} J_\mathrm{E} + \sigma_0 \left(E + \frac{\mu}{T} \nabla T \right) \tag{13}$$

where E is the external electric field, ∇T is the external temperature gradient, W is the enthalpy, and J_E is the energy current. The momentum balance equation becomes

$$-enE - s\nabla T = \Gamma \mathcal{P} \tag{14}$$

Here Γ is the momentum relaxation rate, which in graphene comes from electron-impurity scattering at low temperature.

(f) Find expressions for σ, α, $\bar{\kappa}$ and κ in the "hydrodynamic" theory. You can use the identity

$$W = -en\mu + Ts \tag{15}$$

to simplify your answers.

(g) If impurity scattering is very weak, can this "hydrodynamic" theory for transport be distinguished from the bipolar diffusion theory?