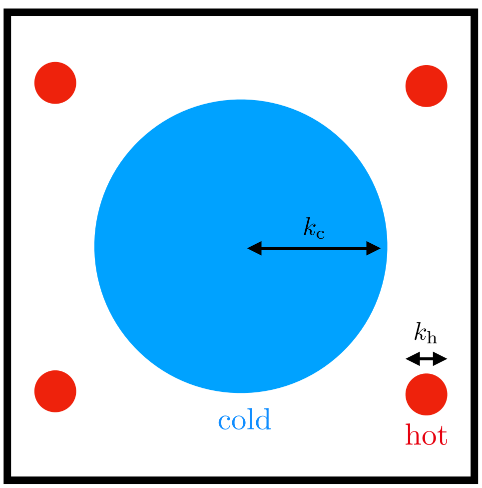
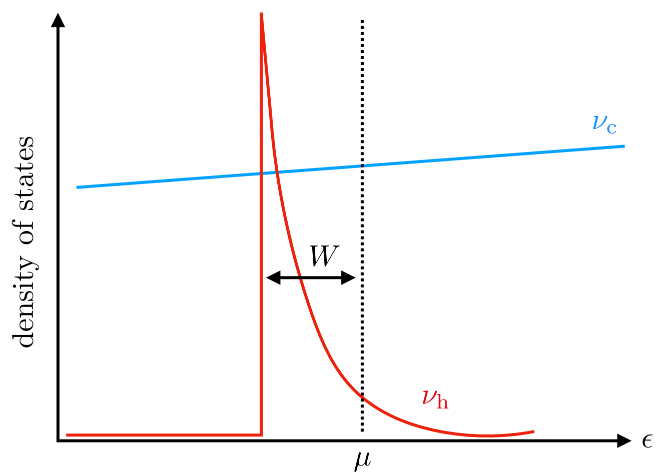
# Homework 2

**Due:** 3:00 PM, Friday, October 4.

**Problem 1 (Hot and cold fermions):** A toy model for certain "strange metals" with unusual $T$-linear resistivity at low temperatures, including $Sr_3Ru_2O_7$, is a weakly interacting electron gas with the following unusual band structure, consisting of a large "cold" Fermi surface along with some small "hot" Fermi surfaces, as shown in the figure. While the figure may appear two dimensional, you should consider the hot and cold Fermi surfaces as (approximately) spheres of radius $k_\mathrm{h}$ and $k_\mathrm{c}$ respectively in the $d$-dimensional Brillouin zone.




It is possible for the hot density of states to have a very sharp cusp, as shown in the figure, if the hot band has a very flat minimum.

The purpose of this problem is to estimate the electron-electron scattering contribution to the resistivity of a metal with this Fermi surface. As this calculation gets very involved rather quickly, you should make a number of simplifying assumptions. Firstly, neglect all O(1) coefficients and set $\hbar = 1$. Secondly, assume the electron-electron collision integral is structureless:

$$\begin{aligned}
\langle \Phi|\mathsf{W}|\Phi\rangle &\sim \frac{1}{T}\int \mathrm{d}^d\mathbf{p}_1\mathrm{d}^d\mathbf{p}_2\mathrm{d}^d\mathbf{p}_3\mathrm{d}^d\mathbf{p}_4\, \delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\delta(\epsilon(\mathbf{p}_1)+\epsilon(\mathbf{p}_2)-\epsilon(\mathbf{p}_3)-\epsilon(\mathbf{p}_4)) \\
&\quad \times (\Phi(\mathbf{p}_1)+\Phi(\mathbf{p}_2)-\Phi(\mathbf{p}_3)-\Phi(\mathbf{p}_4))^2\lambda^2 f_\mathrm{F}(\mathbf{p}_1)f_\mathrm{F}(\mathbf{p}_2)(1-f_\mathrm{F}(\mathbf{p}_3))(1-f_\mathrm{F}(\mathbf{p}_4)) \\
&\sim \frac{\lambda^2}{T}\sum_{\mathbf{b}}\int \mathrm{d}^d\mathbf{p}_1\mathrm{d}^d\mathbf{p}_2\mathrm{d}^d\mathbf{q}\mathrm{d}\omega\, \delta(\epsilon(\mathbf{p}_1)-\epsilon(\mathbf{p}_1+\mathbf{q})+\omega)\delta(\epsilon(\mathbf{p}_2)-\epsilon(\mathbf{p}_2-\mathbf{q}+\mathbf{b})-\omega) \\
&\quad \times (\Phi(\mathbf{p}_1)+\Phi(\mathbf{p}_2)-\Phi(\mathbf{p}_3)-\Phi(\mathbf{p}_4))^2 f_\mathrm{F}(\mathbf{p}_1)f_\mathrm{F}(\mathbf{p}_2)(1-f_\mathrm{F}(\mathbf{p}_1+\mathbf{q}))(1-f_\mathrm{F}(\mathbf{p}_2-\mathbf{q}+\mathbf{b})). \quad (1)
\end{aligned}$$

The momentum integral above runs over the whole Brillouin zone, thus including both cold and hot fermions. Thirdly, approximate Fermi functions with step functions:

$$f_\mathrm{F}(\epsilon) \sim \Theta(\mu+T-\epsilon), \quad (2a)$$
$$1-f_\mathrm{F}(\epsilon) \sim \Theta(\epsilon-\mu+T). \quad (2b)$$

Fourth, in integrals you may always estimate that the group velocity of a cold fermion is $\sim v_c$, the velocity of a hot fermions is $\sim v_h$. Lastly, you may estimate $k_c \gg k_h$, $v_c \gg v_h$, $W \approx v_h k_h$, and that most of the hot fermions lie below the chemical potential. It may be useful to define $\widetilde{W} = \min(T, W)$ for the manipulations below.

(a) First, estimate the resistivity arising from $c + c \to c + c$ scattering (i.e. electrons scattering on the cold Fermi surface). Show that

$$\rho \sim \frac{\lambda^2 T^2 k_c^{d-2}}{v_c^4} \tag{3}$$

The calculation is similar to what is in the lecture notes, but you need to estimate all integrals, up to dimensionless constants.

(b) Next, estimate the resistivity arising from $c + c \to c + h$ scattering; show that

$$\rho \sim \frac{\lambda^2 T \widetilde{W} k_h^{d-1}}{k_c v_c^3 v_h} \tag{4}$$

Is it possible for this contribution to be larger than the contribution arising from only the cold fermions?

(c) Argue that the remaining scattering processes (involving 2 or more hot electrons as either ingoing or outgoing states) can be neglected when estimating resistivity, because they are either generically momentum conserving processes, or because they involve particles on only small portions of the Fermi surface. Use the variational principle with clever trial functions to show that either kind of scattering process can essentially be ignored when estimating the resistivity.

**Problem 2 (Electronic scattering in graphene):** In some metals, the band structure already leads to non-trivial structure in the electron-electron collision integral. As an example, consider the low energy effective description of graphene, which has a low energy effective dispersion relation

$$\epsilon(\mathbf{p}) = \pm v_F |\mathbf{p}| \tag{5}$$

For this problem, you should neglect all momentum dependence in the interacting terms in the electronic Hamiltonian, as in (1), and also ignore umklapp processes. It is conventional in graphene to define a hole as the *absence* of an electron in one of the $\epsilon < 0$ states.

(a) First, consider two electrons scattering off of each other. Show that in two dimensions, with the relativistic dispersion relation, the collision integral $\langle \Phi | \mathsf{W} | \Phi \rangle$ diverges due to collisions between two electrons whose momenta $\mathbf{p}_{1,2}$ are nearly parallel. This is called a **collinear scattering singularity**, and must be resolved via "non-perturbative" screening effects.

(b) It is natural to propose a scattering process – **pair creation** – in which an electron spontaneously creates two particles: one electron and one hole. Keeping in mind what we said a hole was previously, sketch the band structure of graphene along with the initial and final states of two physical quasiparticles which would correspond to the pair creation process. Then argue that due to the relativistic dispersion relation, the rate of pair creation is formally zero if the collision integral only consists of two body scattering.

(c) Guess the qualitative structure of a 3-body contribution to the collision integral, neglecting the dependence of scattering rates on incoming/outgoing momenta. Then argue that if the Hamiltonian had 3-body scattering ($H = \epsilon c^\dagger c + U' c^\dagger c^\dagger c^\dagger ccc$, for example), the rate of pair creation would be finite.

**Problem 3 (Thermal conductivity of amorphous solids):** Consider the propagation of long wavelength acoustic phonons of wave number $q$ in an amorphous solid, which is highly disordered on extremely short length scales. This leads to the following schematic form for the phonon-impurity scattering time:

$$\tau_{\mathrm{imp}}(\mathbf{p}) \sim \frac{1}{p_0^2} + \frac{1}{|\mathbf{p}|^2} \tag{6}$$

where $p_0$ is a constant. Sketch the thermal conductivity as a function of temperature $T$.

**Problem 4 (High temperature resistivity of semiconductors):** Consider the following model for a semiconductor: there is a single electron band with dispersion

$$\epsilon(\mathbf{k}) = \frac{|\mathbf{k}|^2}{2m}. \tag{7}$$

The chemical potential $\mu$ is taken to be comparable to temperature $T$, such that the Fermi surface is significantly smeared out. Consider an acoustic phonon with velocity $v_{\mathrm{ph}} \ll v_{\mathrm{F}} = \sqrt{2\mu/m}$, and consider the usual electron-phonon scattering process $\mathrm{e} \leftrightarrow \mathrm{e} + \mathrm{p}$, with a collision integral analogous to the one used in the lecture notes.

(a) Argue that the energy carried by the phonon will be extremely small. Conclude that the phonon collision integral can thus be approximated by the *impurity* collision integral, with an effective impurity scattering time $\tau \sim T^{-1}$.

(b) What is the temperature dependence of electrical resistivity due to electron-phonon scattering?