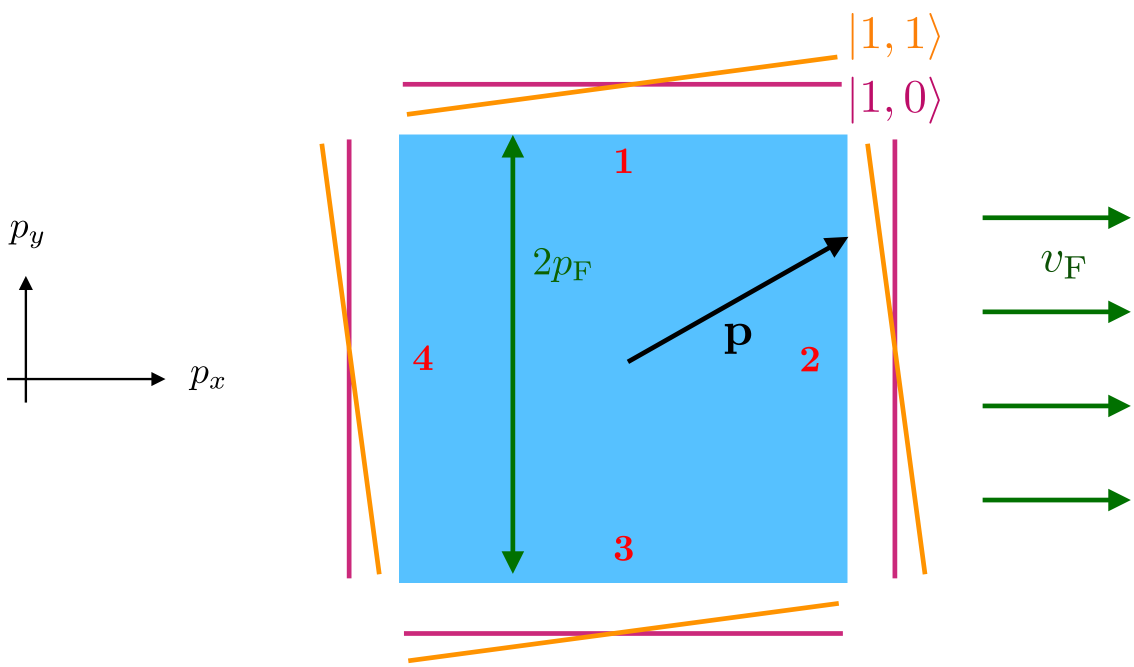
Homework 3

Due: 3:00 PM, Friday, October 18.

Problem 1 (Square Fermi surface): Consider a two-dimensional Fermi liquid whose Fermi surface forms an (approximately) perfect square, as depicted below:



The red labels $m = 1, 2, 3, 4$ in the figure denote the *side* of the Fermi surface that an electron is on. Each side of the Fermi surface has a length of $2p_F$ in momentum space. We assume that the Fermi velocity of quasiparticles has magnitude v_F at every point on the Fermi surface, and that the Fermi surface is electron-like. Hence, the Fermi velocity points perpendicular to each side of the square, as shown for side 2 in the figure.

As in the lecture notes, we will develop a linearized kinetic theory describing the quasiparticles on this Fermi surface. We write

$$|\Phi\rangle = \int \mathrm{d}^d\mathbf{p}\, \Phi(\mathbf{p})|\mathbf{p}\rangle, \tag{1}$$

where we employ the kinetic inner product for vectors $|\mathbf{p}\rangle$. We parameterize the points on the Fermi surface by a discrete side index m, and a parameter s obeying $|s| \le p_F$, in the following way:

$$\begin{cases} m=1 & \mathbf{p}(m,s) = s\hat{\mathbf{x}} + p_F\hat{\mathbf{y}} \\ m=2 & \mathbf{p}(m,s) = p_F\hat{\mathbf{x}} - s\hat{\mathbf{y}} \\ m=3 & \mathbf{p}(m,s) = -s\hat{\mathbf{x}} - p_F\hat{\mathbf{y}} \\ m=4 & \mathbf{p}(m,s) = -p_F\hat{\mathbf{x}} + s\hat{\mathbf{y}} \end{cases}. \tag{2}$$

(a) Show that at temperatures $T \ll \mu$, the kinetic inner product of any two vectors $\langle \Phi_1 | \Phi_2 \rangle$ only depends on the values of $\Phi_{1,2}(\mathbf{p})$ at a point on the Fermi surface parameterized by (2). Hence argue that

$$|\Phi\rangle \approx \sum_{m=1}^{4} \int \mathrm{d}s \; \Phi_m(s) |m, s\rangle \tag{3}$$

where $\Phi_m(s) = \Phi(\mathbf{p}(m, s))$. Show that

$$\langle m', s' | m, s \rangle = C \delta_{mm'} \delta(s - s'). \tag{4}$$

You do not need to determine the proportionality constant C in (4) at this time.

(b) From the form of (4), a natural set of orthogonal basis vectors comes from the Legendre polynomials

$$|m, n\rangle = \int_{-p_\mathrm{F}}^{p_\mathrm{F}} \mathrm{d}s \; \mathrm{L}_n(s) |m, s\rangle, \tag{5}$$

where $\mathrm{L}_0(s) = 1$ and $\mathrm{L}_1(s) = s$ are the first (and only) two polynomials we will need in this problem. From (4), we conclude that $|m, n\rangle$ form an orthogonal basis set. Explain the following formulas for the abstract vectors for charge density $|\mathsf{n}\rangle$, momentum density $|\mathsf{P}_i\rangle$, charge current $|\mathsf{J}_i\rangle$ and stress tensor $|\mathsf{T}_{ij}\rangle$:

$$|\mathsf{n}\rangle = -e(|1,0\rangle + |2,0\rangle + |3,0\rangle + |4,0\rangle), \tag{6a}$$
$$|\mathsf{P}_x\rangle = |1,1\rangle + p_\mathrm{F}|2,0\rangle - |3,1\rangle - p_\mathrm{F}|4,0\rangle, \tag{6b}$$
$$|\mathsf{P}_y\rangle = p_\mathrm{F}|1,0\rangle - |2,1\rangle - p_\mathrm{F}|3,0\rangle + |4,1\rangle, \tag{6c}$$
$$|\mathsf{J}_x\rangle = -ev_\mathrm{F}(|2,0\rangle - |4,0\rangle), \tag{6d}$$
$$|\mathsf{J}_y\rangle = -ev_\mathrm{F}(|1,0\rangle - |3,0\rangle), \tag{6e}$$
$$|\mathsf{T}_{xx}\rangle = p_\mathrm{F}v_\mathrm{F}(|2,0\rangle + |4,0\rangle), \tag{6f}$$
$$|\mathsf{T}_{xy}\rangle = v_\mathrm{F}(|1,1\rangle + |3,1\rangle), \tag{6g}$$
$$|\mathsf{T}_{yx}\rangle = -v_\mathrm{F}(|4,1\rangle + |2,1\rangle), \tag{6h}$$
$$|\mathsf{T}_{yy}\rangle = p_\mathrm{F}v_\mathrm{F}(|1,0\rangle + |3,0\rangle). \tag{6i}$$

Here and/or below, you may want to use `Mathematica` for symbolic manipulations. The qualitative shape of the $|m, 0\rangle$ and $|m, 1\rangle$ basis vectors is depicted in the figure, and may be helpful.

(c) Use $e^2\nu = \langle \mathsf{n} | \mathsf{n} \rangle$ to fix the proportionality coefficient C of (4) in terms of e, ν, p_F and v_F.

(d) Find $\mathbf{v} \cdot \nabla_\mathbf{x} |m, n\rangle$ for any m and n. Remember that the $\nabla_\mathbf{x}$ "passes through" the basis vectors $|m, n\rangle$ which do not depend on spatial position.

(e) Assume the following relaxation time approximation for the collision integral:

$$\mathsf{W} = \frac{1}{\tau}\left(1 - \frac{|\mathsf{n}\rangle\langle\mathsf{n}|}{\langle\mathsf{n}|\mathsf{n}\rangle} - \frac{|\mathsf{P}_x\rangle\langle\mathsf{P}_x|}{\langle\mathsf{P}_x|\mathsf{P}_x\rangle} - \frac{|\mathsf{P}_y\rangle\langle\mathsf{P}_y|}{\langle\mathsf{P}_y|\mathsf{P}_y\rangle}\right), \tag{7}$$

where $\tau \sim T^{-2}$. Show that for any $|\Phi\rangle$, if $\mathsf{W}^{-1}|\Phi\rangle$ exists, then

$$\mathsf{W}^{-1}|\Phi\rangle = \tau|\Phi\rangle \tag{8}$$

and

$$\langle \Phi | \mathsf{n} \rangle = \langle \Phi | \mathsf{P}_x \rangle = \langle \Phi | \mathsf{P}_y \rangle = 0. \tag{9}$$

(f) Show that the incoherent conductivity σ_0 of this Fermi liquid scales as $\sigma_0 \sim T^{-2}$. Compare this result to what we found in the lecture notes for the isotropic Fermi liquid and comment on any discrepancy.

(g) On symmetry grounds, show that the most general possible viscosity tensor for this model is

$$\begin{pmatrix} \eta_{xxxx} & \eta_{xxxy} & \eta_{xxyx} & \eta_{xxyy} \\ \eta_{xyxx} & \eta_{xyxy} & \eta_{xyyx} & \eta_{xyyy} \\ \eta_{yxxx} & \eta_{yxxy} & \eta_{yxyx} & \eta_{yxyy} \\ \eta_{yyxx} & \eta_{yyxy} & \eta_{yyyx} & \eta_{yyyy} \end{pmatrix} = \begin{pmatrix} \eta_1 & 0 & 0 & \eta_2 \\ 0 & \eta_3 & \eta_4 & 0 \\ 0 & \eta_4 & \eta_3 & 0 \\ \eta_2 & 0 & 0 & \eta_1 \end{pmatrix} \tag{10}$$

and that all 4 coefficients above could be distinct. In an isotropic fluid, what would $\eta_{1,2,3,4}$ be as functions of the shear viscosity η and bulk viscosity ζ? Explain what symmetries have been broken by the square Fermi surface which allow for two new coefficients.

(h) Calculate the 16 components of the viscosity tensor η_{ijkl}. Confirm that this explicit kinetic model is consistent with (10), and that the explicit η_{ijkl} is positive semidefinite, when interpreted as a 4×4 matrix as in (10).

Problem 2 (Hydrodynamic plasmon decay with incoherent conductivity): Consider the equations

$$0 = \nu \partial_t \delta\mu(\mathbf{x},t) + n_0 \partial_i \delta u_i(\mathbf{x},t) - \sigma_0 \partial_i \partial_i \left(\delta\mu(\mathbf{x},t) + \nu \int \mathrm{d}^d\mathbf{y}\, K(\mathbf{x}-\mathbf{y}) \delta\mu(\mathbf{y},t) \right), \tag{11a}$$

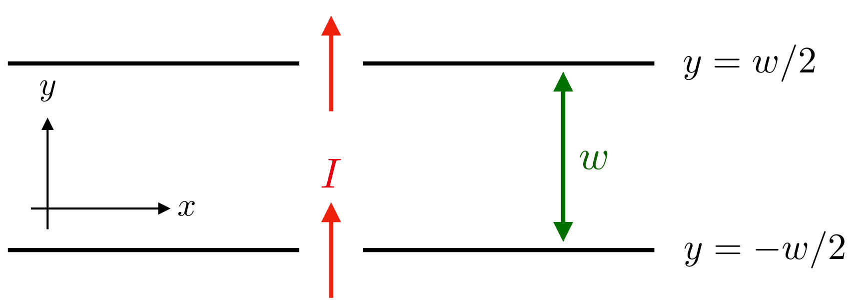
$$\begin{aligned} 0 = {} & m n_0 \partial_t \delta u_i(\mathbf{x},t) + n_0 \partial_i \left(\delta\mu(\mathbf{x},t) + \nu \int \mathrm{d}^d\mathbf{y}\, K(\mathbf{x}-\mathbf{y}) \delta\mu(\mathbf{y},t) \right) \\ & - \eta \partial_j \partial_j \delta u_i(\mathbf{x},t) - \left(\zeta + \frac{d-2}{d}\eta \right) \partial_j \partial_i \delta u_j(\mathbf{x},t) \end{aligned} \tag{11b}$$

governing the propagation of hydrodynamic plasmons in a low temperature Fermi liquid with an incoherent charge conductivity. All parameters except for the Coulomb kernel

$$K(\mathbf{x}) = \frac{\alpha}{|\mathbf{x}|} \tag{12}$$

are to be treated as constants. Find the leading order contributions to both the real and imaginary parts of the dispersion relation $\omega(k)$ of the propagating plasmon mode in both $d = 2$ and $d = 3$ spatial dimensions. Comment on the differences and similarities between your answer and the answer found in class for plasmons propagating in an approximately Galilean-invariant system.

Problem 3 (Whirlpools of viscous electrons): Consider the geometry shown below:



In this problem, you may assume that the boundary conditions are "no slip" – namely, $v_x = 0$ when $|y| = w/2$ and $x \neq 0$.

(a) Starting from the quasihydrodynamic equations in the "Galilean" Fermi liquid with weak momentum relaxation in the lecture notes, argue that

$$u_i = \epsilon_{ij}\partial_j\psi \tag{13}$$

for a scalar function ψ called the **stream function**, and that

$$\partial_i\partial_i\left(\partial_j\partial_j - \frac{1}{\lambda^2}\right)\psi = 0, \tag{14}$$

where λ is the Gurzhi length.

(b) Suppose that the velocity profile $v_y(x, -w/2) = v_y(x, w/2) = f(x)$ is specified. Solve (14) with the specified boundary conditions. You should do so by taking the Fourier transform of ψ and f in the x-direction alone, and obtain

$$\psi(k, y) = -\frac{f(k)}{\mathrm{i}k}\frac{\cosh ky - \dfrac{k\lambda}{\sqrt{k^2\lambda^2+1}}\sinh\dfrac{kw}{2}\operatorname{csch}\dfrac{\sqrt{k^2\lambda^2+1}w}{2\lambda}\cosh\dfrac{\sqrt{k^2\lambda^2+1}y}{\lambda}}{\cosh\dfrac{kw}{2} - \dfrac{k\lambda}{\sqrt{k^2\lambda^2+1}}\sinh\dfrac{kw}{2}\coth\dfrac{\sqrt{k^2\lambda^2+1}w}{2\lambda}}. \tag{15}$$

(c) Using `Mathematica` or other software, and assuming that

$$f(x) = \frac{I}{-en_0}\delta(x), \tag{16}$$

where I represents the total charge current flowing in and out of the device, make a surface/color plot of the velocity profiles $u_x(x, y)$ and $u_y(x, y)$ in the channel. I would suggest focusing on the region $x \sim w$ – you may need to adjust the color scheme to see interesting physics arise. Comment on the qualitative behavior of the solution as a function of the dimensionless ratio w/λ. (Think of the problem title.) If you had an experimental set-up (such as scanning SQuID microscopy) where you could locally image the electric current (and thus velocity), how would you detect the transition from Ohmic, diffusive flow of current to viscous flow?