Homework 4

Due: 3:00 PM, Friday, November 1.

Problem 1 (Variational principle for hydrodynamic transport): Consider a hydrodynamic theory for a Galilean-invariant Fermi liquid. The hydrodynamic transport equations read

$$\partial_i J_i = 0, \tag{1a}$$

$$\partial_i Q_i = 0, \tag{1b}$$

$$n\partial_i\mu + s\partial_i T - \partial_j\left(\eta\left[\partial_j u_i + \partial_i u_j - \frac{2}{d}\delta_{ij}\partial_k u_k\right] + \zeta\delta_{ij}\partial_k u_k\right) = -enE_i - s\partial_i T_{\text{ext}} \tag{1c}$$

where the charge and heat current are

$$J_i = -enu_i, \tag{2a}$$

$$Q_i = T_0 s u_i - \kappa_0(\partial_i T + \partial_i T_{\text{ext}}) \tag{2b}$$

Here E_i and $-\partial_i T_{\text{ext}}$ denote the *small* external electric fields and temperature gradients used to drive thermoelectric transport, while μ, T and u_i denote the *small* perturbations of the fluid away from equilibrium in response to the sources. Suppose that the parameters κ_0, η, n and s may vary in space, but the equilibrium temperature T_0 does not vary with position. Define the functional

$$\mathcal{R}[J_i, Q_i] = \int \frac{\mathrm{d}^d\mathbf{x}}{V}\left[\frac{\eta}{2e^2}\left(\partial_i\frac{J_j}{n} + \partial_j\frac{J_i}{n} - \frac{2}{d}\delta_{ij}\partial_k\frac{J_k}{n}\right)^2 + \frac{\zeta}{e^2}\left(\partial_k\frac{J_k}{n}\right)^2 + \frac{1}{T_0\kappa_0}\left(Q_i + \frac{T_0 s}{en}J_i\right)^2\right]. \tag{3}$$

where V represents the volume of space.

(a) Suppose we happened to know the profile of the charge current J_i and the heat current Q_i, as a function of the sources E_i and $-\partial_i T_{\text{ext}}$, on the exact solution to (1). Call these special profiles $J_i = J_i^*$ and $Q_i = Q_i^*$. Show that

$$\mathcal{R}[J_i, Q_i] = \begin{pmatrix} \overline{J}_i & \overline{Q}_i \end{pmatrix}\begin{pmatrix} \sigma & T_0\alpha \\ T_0\tilde{\alpha} & T_0\bar{\kappa} \end{pmatrix}^{-1}_{ij}\begin{pmatrix} \overline{J}_j \\ \overline{Q}_j \end{pmatrix} \tag{4}$$

where

$$\overline{J}_i = \int \frac{\mathrm{d}^d\mathbf{x}}{V} J_i, \tag{5a}$$

$$\overline{Q}_i = \int \frac{\mathrm{d}^d\mathbf{x}}{V} Q_i, \tag{5b}$$

and the thermoelectric conductivity matrix is given by

$$\begin{pmatrix} \overline{J}_i \\ \overline{Q}_i \end{pmatrix} = \begin{pmatrix} \sigma & T_0\alpha \\ T_0\tilde{\alpha} & T_0\bar{\kappa} \end{pmatrix}_{ij}\begin{pmatrix} E_j \\ -T_0^{-1}\partial_j T_{\text{ext}} \end{pmatrix} \tag{6}$$

Hint: Begin by multiplying (1c) by u_i and integrating over space.

(b) Now consider $\mathcal{R}[J_i, Q_i]$ as a function of *arbitrary* currents J_i and Q_i subject to the constraints (1a), (1b) and (5). Prove that

$$\mathcal{R}[J_i^*, Q_i^*] \le \mathcal{R}[J_i, Q_i]. \tag{7}$$

Give a physical interpretation to this variational principle.
Hint: Proceed by mirroring our proof of the variational principle in the kinetic theory of transport, writing $J_i = J_i^* + \widetilde{J}_i$ and $Q_i = Q_i^* + \widetilde{Q}_i$. What are the constraints on $\widetilde{J}_i$ and $\widetilde{Q}_i$? Expand out $\mathcal{R}[J_i, Q_i]$ and show that similar manipulations to the previous part imply that the linear terms in deviations $\widetilde{J}_i$ and $\widetilde{Q}_i$ vanish.

(c) Plug in the ansatz

$$J_i = -en_0 u_i^0, \tag{8a}$$
$$Q_i = T_0 s_0 u_i^0 \tag{8b}$$

with u_i^0 a constant vector to this variational principle. Take n_0 and s_0 to be the average values of n and s respectively. Compare $\mathcal{R}$ to the perturbative result for ρ_{ij} in the lecture notes, and comment on any similarities.

(d) Use this variational principle to exactly solve the hydrodynamic transport problem in a fluid in one spatial dimension ($d = 1$): i.e., solve for σ, α and $\bar{\kappa}$ as arbitrary functions of ζ, n and s.

Problem 2 (Long lived quadrupolar fluctuations): Suppose that we modify our relaxation time model of a two dimensional isotropic Fermi liquid, such that the collision integral is

$$\mathsf{W}|m\rangle = \begin{cases} 0 & |m| \le 1 \\ \gamma'|m\rangle & |m| = 2 \\ \gamma|m\rangle & |m| > 2 \end{cases}. \tag{9}$$

Here $|m\rangle$ represent the angular harmonic fluctuations $\Phi = \mathrm{e}^{\mathrm{i}m\theta}$, as discussed in the lecture notes.

(a) Following the derivation in the lecture notes when $\gamma' = \gamma$, show that

$$\langle 0|(\mathsf{W} + \mathrm{i}\mathbf{k}\cdot\mathbf{v})^{-1}|0\rangle = \frac{\nu}{\sqrt{\gamma^2 + (kv_\mathrm{F})^2} + 2\gamma' - \gamma} \tag{10}$$

You may find it useful to use `Mathematica` to do symbolic matrix inversion. Here ν represents the density of states of the Fermi liquid.

(b) Give a heuristic sketch of the temperature dependence of the resistivity in a metal with long wavelength inhomogeneity on the length scale ξ. Assume the disorder is perturbatively weak. Your sketch should depend on the ratio γ'/γ.

Problem 3 (Terahertz radiation): Consider an electron liquid in a one dimensional channel of length L, parameterized by $0 \le x \le L$. Beyond linear response, assume that the ideal hydrodynamic equations are

$$\partial_t n + \partial_x(nu) = 0, \tag{11a}$$
$$\partial_t(mnu) + \partial_x\left(mnu^2\right) + mv_\mathrm{s}^2\partial_x n = 0. \tag{11b}$$

Here m, v_s and η are constants corresponding to effective mass, speed of sound and shear viscosity, respectively.

(a) Show that any uniform density $n(x,t) = n_0$ and any uniform velocity $u(x,t) = u_0$ solve the nonlinear equations above.

(b) For the moment, set $\eta = 0$. Suppose that the boundary conditions on the fluid are

$$n(x=0,t) = n_0, \tag{12a}$$
$$n(x=L,t) \times u(x=L,t) = n_0 u_0. \tag{12b}$$

Now let us look at the stability of the homogeneous solution. Let

$$n(x,t) = n_0 + \delta n(x,t), \tag{13a}$$
$$u(x,t) = u_0 + \delta u(x,t). \tag{13b}$$

Expand the hydrodynamic equations to linear order in perturbations, and determine the boundary conditions on δn and δu.

(c) Assuming that $0 < u_0 < v_\mathrm{s}$, show that the normal modes of the cavity (i.e. solutions which have time dependence $\delta n(x,t) = \delta n(x)\mathrm{e}^{-\mathrm{i}\omega t}$, and $\delta u(x,t) = \delta u(x)\mathrm{e}^{-\mathrm{i}\omega t}$) have complex frequency ω given by

$$\omega_n = \pm\frac{\pi n\left(v_\mathrm{s}^2 - u_0^2\right)}{2Lv_\mathrm{s}} + \mathrm{i}\frac{v_\mathrm{s}^2 - u_0^2}{2Lv_\mathrm{s}}\log\frac{v_\mathrm{s}+u_0}{v_\mathrm{s}-u_0}, \quad (n = 1,3,5,\ldots). \tag{14}$$

(d) In a proposed device using an electron liquid in graphene, $L = 10^{-6}$ m and $v_\mathrm{s} = 10^6$ m/s. Show that the frequency of the hydrodynamic oscillations above is ~ 1 THz. This is a notoriously challenging frequency of electromagnetic radiation to generate, and it has been proposed that the hydrodynamic instability above could drive a coherent source of THz radiation.

If the decay rate of propagating sound waves is $\Gamma_\mathrm{s}(k,\omega)$, then (14) is approximately modified to

$$\omega_n = \pm\frac{\pi n\left(v_\mathrm{s}^2 - u_0^2\right)}{2Lv_\mathrm{s}} + \mathrm{i}\frac{v_\mathrm{s}^2 - u_0^2}{2Lv_\mathrm{s}}\log\frac{v_\mathrm{s}+u_0}{v_\mathrm{s}-u_0} - \mathrm{i}\mathit{\Gamma}_\mathrm{s}(k_n) \tag{15}$$

where k_n is an appropriate wave number for harmonic n.

(e) Argue that if τ_ee is a momentum-conserving scattering rate between electrons, that the approximate functional form

$$\mathit{\Gamma}_\mathrm{s}(k) = \frac{v_\mathrm{F}^2}{8}\frac{k^2\tau_\mathrm{ee}}{1+\frac{1}{4}(\tau_\mathrm{ee}v_\mathrm{s}k)^2} \tag{16}$$

qualitatively interpolates between the known decay rates of first sound waves and zero sound waves. Use the known hydrodynamic coefficients for the isotropic two dimensional Fermi liquid in the relaxation time model discussed in the lecture notes. Then combine (15) and (16) and *sketch* the region in the (u_0, τ_ee) plane where the instability exists.

(f) Now suppose that there is momentum relaxing scattering that occurs on the time scale τ_imp. Following the discussion in the lecture notes on quasihydrodynamic sound modes, estimate the contribution of this momentum relaxing scattering to $\mathit{\Gamma}_\mathrm{s}$. Then, estimate the minimal value of the mean free path for momentum-relaxing collisions, below which in the graphene device above it would be impossible to observe an instability. Compare your result to the standard estimated mean free path of 10^{-6} m in a reasonable quality graphene device.