Homework 5

Due: 3:00 PM, Friday, November 22.

Problem 1 (Collisionless magnetotransport in two dimensions): In this problem, we will study the Hall conductivity of two dimensional metals in the limit where there is no scattering – each quasiparticle only moves in the background magnetic field along the Fermi surface. The tool we will use is Chambers' formula:

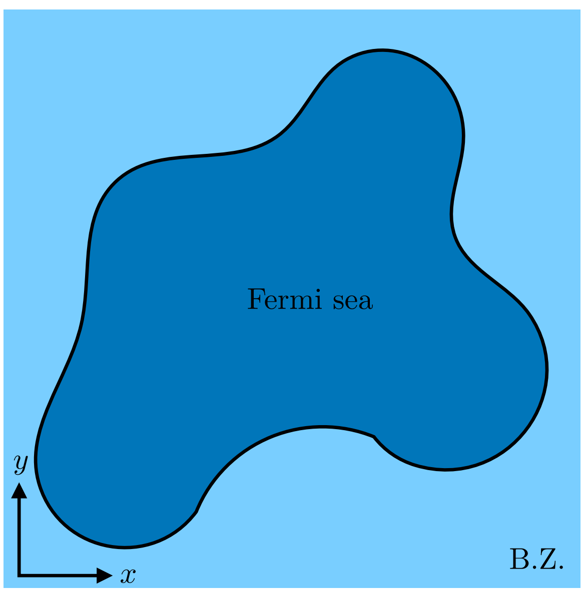
$$\sigma_{ij} = e^2 \int \frac{\mathrm{d}^2\mathbf{p}}{(2\pi\hbar)^2} \left(-\frac{\partial f_\mathrm{F}}{\partial \epsilon}\right) \int_{-\infty}^{0} \mathrm{d}t\ \mathrm{e}^{t/\tau} v_i(\mathbf{p}) v_j(\mathbf{p}(t)). \tag{1}$$

Here τ is a regulator; the limit $\tau \to \infty$ must always be taken at the end of a calculation, and $\mathbf{p}(t)$ solves

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -eB\epsilon_{ij} v_j(\mathbf{p}). \tag{2}$$

with the boundary condition on the final time $\mathbf{p}(0) = \mathbf{p}$. In this problem, we take the low temperature limit $T \to 0$, so the Fermi surface is well defined.

Begin by assuming the Fermi surface consists of a single **closed** contour, as in the figure below. The occupied electronic states are more darkly shaded.

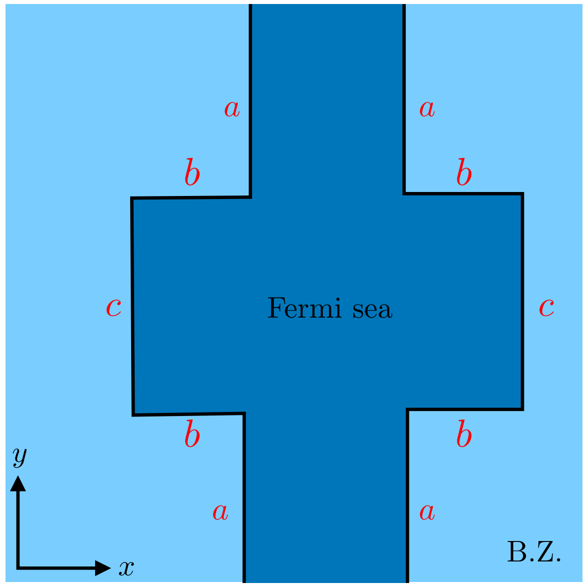


(a) Show that

$$\sigma_{ij} = -\frac{en}{B}\epsilon_{ij}, \tag{3}$$

and justify this answer on physical grounds, without resorting to (1).
Hint: Start by evaluating the t integral in (1), using (2). You should take $\tau \to \infty$.

Next, consider the following **open** Fermi surface. The lengths in the Brillouin zone of each segment are shown; assume that each segment is oriented exactly along either the x or the y direction, as depicted. Assume the local density of states is ν everywhere on the Fermi surface, and that the magnitude of the Fermi velocity is v_F everywhere along the Fermi surface.



(b) Using (1), calculate σ_{xy} for the metal described above. Show that

$$\sigma_{xy} = \frac{2\nu e v_\mathrm{F}}{B}\frac{b^2(2a-c)}{2a+2b+c} \tag{4}$$

(c) If an experimentalist measures σ_{xy}, would they measure the number of electrons in the metal? That is, does (3) hold for σ_{xy}? Why or why not?

Problem 2 (Hall viscosity and fluid flows): Consider a fluid in the absence of an external magnetic field or momentum relaxation, but with a non-vanishing Hall viscosity. Such a fluid could arise, for example, in a thin film of chiral liquid crystals.

(a) Write down the coupled hydrodynamic equations for number/mass and momentum conservation within linear response about equilibrium $n = n_0$ and $u_i = 0$, assuming time-independent solutions. Show that you may write

$$u_i = \epsilon_{ij}\partial_j\psi \tag{5}$$

and that the linearized hydrodynamic equations reduce to

$$\partial_i\partial_i\partial_j\partial_j\psi = 0. \tag{6}$$

Conclude that Hall viscosity can only modify flows by modifying the boundary conditions, and/or can only be measured through its effects at boundaries.

(b) As an example of the effect of Hall viscosity, consider a pressure gradient applied down a long narrow channel of width w. Solve the hydrodynamic equations with no-slip boundary conditions, and assuming no fluid flows through the walls of the channel. Then calculate the forces per unit length acting on each wall of the channel, and comment on what Hall viscosity does.
Hint: we solved a (more complicated) version of this problem in the lecture notes.

Problem 3 (Sound waves in a magnetic field): Neglecting derivative corrections to the currents (i.e. viscosities or incoherent conductivity), write down the linearized hydrodynamic equations for charge and momentum conservation, in the presence of a weak magnetic field. Find the dispersion relations $\omega(k)$ for the quasinormal modes exactly (within these approximations), and describe what happens to sound waves in a magnetic field.

Problem 4 (Spectral weight and work): Consider a many-body system with equilibrium Hamiltonian H_0, weakly perturbed by the operator $\mathcal{O}$ in a time-dependent manner as follows:

$$H(t) = H_0 - h(t)\mathcal{O}. \tag{7}$$

Treat $h(t)$ as perturbatively small, and suppose $h(t)$ vanishes sufficiently far in the past.

(a) At times $t \to -\infty$, $\rho(t) = \rho_0$, the thermal density matrix. Find an expression, valid to linear order in h, for the perturbed density matrix $\rho(t)$ by solving the Schrödinger equation.

(b) Calculate the work done on the system per unit time, as defined by

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}\left(\rho(t)H(t)\right), \tag{8}$$

to order $\mathrm{O}(h^2)$. Now integrate this over time to evaluate the total work performed. Show that the total work W_{tot} performed is

$$W_{\mathrm{tot}} = \int \frac{\mathrm{d}\omega}{2\pi}\, \omega |h(\omega)|^2 \mathrm{Im}\left(G^{\mathrm{R}}_{\mathcal{O}\mathcal{O}}(\omega)\right). \tag{9}$$

Give a physical justification for the positivity constraint $\omega \mathrm{Im}\left(G^{\mathrm{R}}_{\mathcal{O}\mathcal{O}}(\omega)\right) \geq 0$ that we derived in the lectures notes using more rigorous arguments.