Homework 6

Due: 3:00 PM, Friday, December 6.

Problem 1 (Johnson noise): A resistor plugged in to a very sensitive ohmmeter can detect statistical fluctuations in the resistance as a function of time. This effect is called **Johnson noise**, and is easily seen in experiment. Assume that a resistor consists of a metal with an isotropic conductivity tensor, and that the system is at temperature T. The resistor of cross-sectional area A is placed between two leads a distance L apart. Assume that the microscopic dynamics is time reversal invariant.

(a) Experiments measure the current autocorrelation function, which measures the time-resolved fluctuations in the total current. This is captured by the symmetric Green's function $S_{J_xJ_x}(\mathbf{k}=\mathbf{0},\omega)$. Using properties of Green's functions in the lecture notes, show that

$$S_{J_xJ_x}(\omega) = \omega\coth\frac{\beta\omega}{2} \times \mathrm{Re}\left(\sigma_{xx}(\omega)\right). \tag{1}$$

(b) The ohmmeter measures voltage fluctuations in the frequency range $|\omega| < B$. Argue that if $B \to 0$, and $\sigma_{xx}(\omega)$ is approximately independent of ω in this voltage range, then the variance in the voltage in the absence of a bias current is

$$\left\langle V^2\right\rangle_{\mathrm{RMS}} = \int_{-B}^{B} \mathrm{d}\omega \,\left\langle V^2\right\rangle(\omega) = 4k_{\mathrm{B}}TRB. \tag{2}$$

What frequency scale(s) does B need to be small compared to?
Hint: Begin by converting $S_{J_xJ_x}$ into a correlation function for statistical fluctuations of the net current, $\langle I^2\rangle(\omega)$ – this amounts to keeping track of the geometric factors. Then use Ohm's Law.

(c) Estimate $\sqrt{\langle V^2\rangle}$ if $R = 100\ \Omega$, $T = 300$ K and $\frac{1}{2\pi}B = 1$ kHz.

Problem 2 (Sum rules and diffusion): Consider a many-body quantum system with Hamiltonian

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + \sum_i V_1(\mathbf{x}_i) + \sum_{i<j} V_2(\mathbf{x}_i - \mathbf{x}_j), \tag{3}$$

and the number density operator

$$n(\mathbf{x}) = \sum_{i=1}^{N} \delta(\mathbf{x} - \mathbf{x}_i). \tag{4}$$

(a) Starting with the Heisenberg equation of motion for n, show the operator identity

$$\partial_t n + \nabla\cdot\mathbf{J} = 0 \tag{5}$$

where $\mathbf{J}$ is the number current operator

$$\mathbf{J}(\mathbf{x}) = \sum_{i=1}^{N} \left\{\frac{\mathbf{p}_i}{2m}, \delta(\mathbf{x} - \mathbf{x}_i)\right\}. \tag{6}$$

(b) Suppose that in thermal equilibrium $\langle n(\mathbf{x})\rangle = n_0$. Show that the equal time commutator

$$\langle[\mathbf{J}(\mathbf{x}), n(\mathbf{x}')]\rangle = \frac{n_0}{m}\left(\mathrm{i}\nabla_{\mathbf{x}'}\delta(\mathbf{x}-\mathbf{x}')\right). \tag{7}$$

(c) Argue that H is time reversal symmetric. Then, using the ω-integrals of the spectral function from the lecture notes, derive the following **f-sum rule**:

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\pi}\, \omega \mathrm{Im}\left(G^{\mathrm{R}}_{nn}(\mathbf{k},\omega)\right) = \frac{n_0}{m}k^2. \tag{8}$$

These sum rules are one of the few exact results (other than analyticity, etc.) which can be proven for microscopically realistic Hamiltonians. Unfortunately they are usually not very practical, except as a check on numerical results.

Interestingly though, these sum rules have important implications about the hydrodynamic regime.

(d) Consider a hydrodynamic theory whose density $n(\mathbf{x}, t)$ obeys the diffusion equation

$$\partial_t n = D\nabla^2 n \tag{9}$$

in the long wavelength limit. Try to evaluate the sum rule (8) using the Kadanoff-Martin Green's function

$$G^{\mathrm{R}}_{nn}(\mathbf{k},\omega) = \frac{\chi D k^2}{Dk^2 - \mathrm{i}\omega} \tag{10}$$

where χ is the density-density susceptibility; show that the integral is ill-defined.

One way to fix this problem is to modify the charge current to

$$\mathbf{J}(\mathbf{x}, t) = -\int_0^t \mathrm{d}s\, \mathcal{D}(s)\nabla n(\mathbf{x}, t-s). \tag{11}$$

Without loss of generality, take $\mathcal{D}(t)$ to be an even function; assume the dynamics is only defined for positive times $t > 0$ for mathematical convenience. Define $\mathcal{D}(\omega)$ to be the Fourier transform of $\mathcal{D}(t)$, and $\widetilde{\mathcal{D}}(z)$ to be the Laplace transform of $\mathcal{D}(t)$, defined when $\mathrm{Im}(z) > 0$. These are related by a Kramers-Kronig relation:

$$\widetilde{\mathcal{D}}(z) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}}\frac{\mathcal{D}(\omega)}{\omega - z} = \mathcal{D}(z) + \mathrm{i}\mathcal{D}_{\mathrm{P}}(z) \tag{12}$$

$\mathcal{D}_{\mathrm{P}}(z)$ above denotes the principal part of the above integral, and $\mathrm{i}\mathcal{D}_{\mathrm{P}}(z)$ is purely imaginary.

(e) Show that $G^{\mathrm{R}}_{nn}(\mathbf{k}, z)$ is given by the Kadanoff-Martin diffusive Green's function (10), but with the diffusion constant replaced by $\widetilde{\mathcal{D}}(z)$.
Hint: there are nice relations for the Laplace transform of a convolution of functions.

(f) Use the properties of $G^{\mathrm{R}}_{nn}(\mathbf{k},\omega)$, and of the Fourier transform, to show that $\mathcal{D}(\omega)$ is even, real-valued, and non-negative. Conclude that $|\mathcal{D}(t)| \le \mathcal{D}(t = 0)$, and comment on the physical interpretation.

(g) Under what conditions on $\mathcal{D}(t)$ (and/or $\mathcal{D}(\omega)$) does (11) reduce to ordinary diffusion on long time scales? If so, what is the effective diffusion constant?

As a physical model for a time-dependent diffusion constant, consider

$$\partial_t n + \nabla \cdot \mathbf{J} = 0, \tag{13a}$$

$$\partial_t \mathbf{J} + \alpha \nabla n + \mathrm{O}\left(\nabla^2\right) = -\frac{1}{\tau}\mathbf{J}. \tag{13b}$$

(h) Give a quasihydrodynamic interpretation of (13). Evaluate $\mathcal{D}(t)$ in this model along with $G^{\mathrm{R}}_{nn}(\mathbf{k},\omega)$. Show that ordinary diffusion is recovered on long time scales; what is the diffusion constant D?

(i) Show that the sum rule (8) is sensible in the model (13), and that it relates D to τ (along with thermodynamic coefficients).

Problem 3 (Density-density response in a magnetic field): Using the Kadanoff-Martin formalism, calculate the singular/hydrodynamic part of the density-density correlator $G^{\mathrm{R}}_{nn}(\mathbf{k},\omega)$ at small k and ω, for a Galilean-invariant Fermi liquid in a weak magnetic field. For simplicity, you may neglect energy conservation and Hall viscosity. Comment on the limiting forms of this Green's function both as the magnetic field $B \to 0$, and as the shear viscosity $\eta \to 0$.