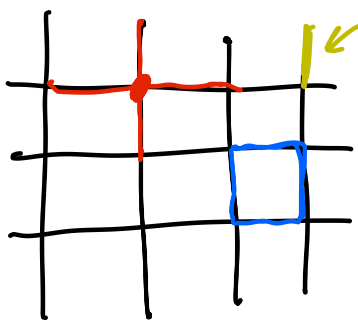
Toric code: $L \times L$ square lattice, periodic BCs:



qubits on edges

X-checks $X^{\partial v}$ on vertices

Z-checks $Z^{\partial f}$ on faces

Today we'll explore the toric code from the perspective of cond-mat $\Rightarrow$ (topologically ordered) phase of matter!

Interpret as $\underline{\mathbb{Z}_2 \text{ lattice gauge theory}}$:

$$H_0 = -\sum_{\text{vertex } v} X^{\partial v} - \sum_{\text{face } f} Z^{\partial f} = -\sum_{v} \prod_{e \sim v} X_e - \sum_{f} \prod_{e \sim f} Z_e$$

To understand this analogy better let's turn to a more understood example of a gauge theory...

Analogy: electromagnetism = $U(1)$ gauge theory

matter free: $\nabla \cdot \vec{E} = \underbrace{\nabla \cdot \vec{B} = 0}$, $\quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, $\quad \nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$

Solve: $\vec{B} = \nabla \times \vec{A}$ and $\vec{E} = -\cancel{\nabla \varphi} - \frac{\partial \vec{A}}{\partial t}$

gauge fix $\varphi = 0$

Set $c=1$ for simplicity:

Lagrangian for E&M: $\mathcal{L} = \frac{1}{2}\vec{E}^2 - \frac{1}{2}\vec{B}^2 = \frac{1}{2}\left(\frac{\partial \vec{A}}{\partial t}\right)^2 - \frac{1}{2}(\nabla \times \vec{A})^2$

$\Rightarrow$ $\vec{E}$ & $\vec{A}$ are conjugate: $[\vec{E}, \vec{A}] = i$

Hamiltonian $H_0 = \int d^3x \left[\frac{1}{2}\vec{E}^2 + \frac{1}{2}(\nabla \times \vec{A})^2\right]$

Lattice gauge theory $\Rightarrow$ define $\vec{A}$ & $\vec{E}$ on edges
(just as in discrete vector calculus before)

Strictly speaking $A_e \sim A_e + 2\pi$ is only a periodic variable. We'll return to this point shortly...

$$H_0 = \sum_{\text{edge } e} \frac{1}{2}\vec{E}_e^{\,2} + \sum_{\text{face } f} \frac{1}{2}\Big(\sum_e \underbrace{\langle f|d_1|e\rangle}_{\text{discretized curl}} A_e\Big)^2$$

(as last time)

But we also need to enforce a constraint: $\underbrace{\nabla\cdot E = 0}$

$$\sum_e \underbrace{\langle e|d_0|v\rangle}_{\text{discretized divergence}} E_e = 0 \quad \text{for each } v$$

Softly implement this constraint:

constraint penalty

$$H_0 = \frac{1}{2}\sum_e E_e^2 + \frac{1}{2}\sum_f\Big(\sum_e \langle f|d_1|e\rangle A_e\Big)^2 + g\sum_v\Big(\sum_e \langle e|d_0|v\rangle E_e\Big)^2$$

First modification: $A_e \sim A_e + 2\pi$ periodic?

$\Rightarrow E_e$ integer-valued

analogy to 2d angular momentum.

replace $[E_e, A_e] = i \Longrightarrow e^{i\phi E_e} e^{il A_e} = e^{il A_e} e^{i\phi E_e} \cdot e^{-il\phi}$

Second modification: restrict to binary ($\mathbb{Z}_2$):

$A_e \in \{0,\pi\}$, $\quad E_e \in \{0,1\}$?

$$e^{iA_e} = Z_e \qquad e^{i\pi E_e} = X_e \qquad \Rightarrow \qquad X_e Z_e = Z_e X_e e^{-i\pi}$$

which is the Pauli algebra

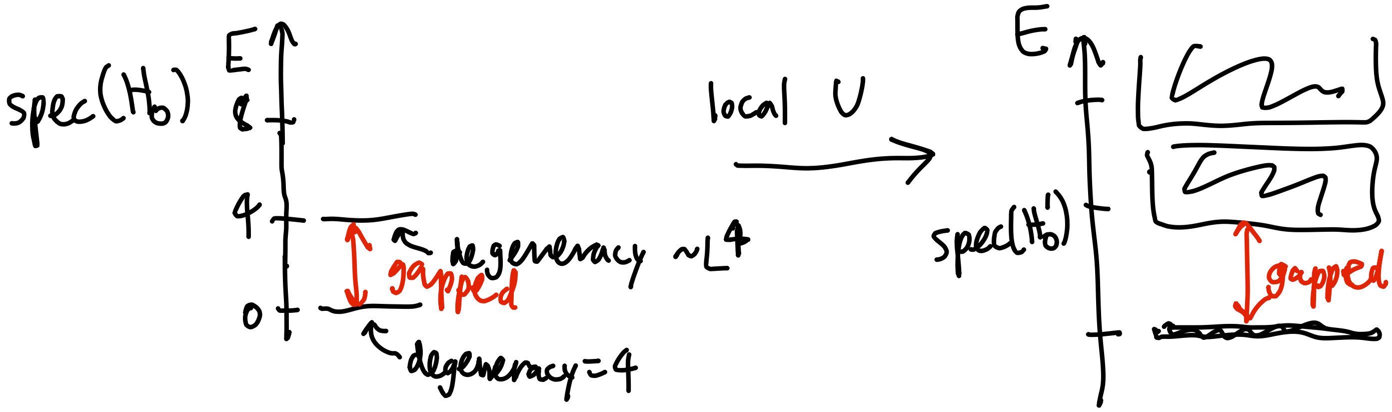
So we get a $\mathbb{Z}_2$ lattice gauge theory.

$$H_0' = -\sum_e \underbrace{h X_e}_{\text{analogous to } E_e^2 \,?} - \underbrace{\sum_f \prod_{e\sim f} Z_e}_{(\nabla\times A)^2} - g\underbrace{\sum_v \prod_{e\sim v} X_e}_{(\nabla\cdot E)^2}$$

This looks like toric code but w/ an extra term? Is that bad?
No! This realizes same phase of matter!

<u>Thm</u>: If $H_0' = H_0 + hV$, for extensive few-body (q-local) V, for small h:

- $\exists$ quasilocal unitary U: $U|\text{g.s. of } H_0\rangle = |\text{l.e.s. of } H_0'\rangle$
- H_0 & H_0' are both gapped.
- degeneracy of H_0' lifted by $\Delta E \lesssim e^{-=} = e^{-\# L}$



Idea behind the proof: Knill-Laflamme conditions in a local sense:

Any <u>local</u> operator $\mathcal{O}_R$ in region R ← size < L

$$\langle \psi_\alpha | \mathcal{O}_R | \psi_\beta \rangle = c_{\mathcal{O}_R} \delta_{\alpha\beta}$$

↑ diff. codewords of H_0

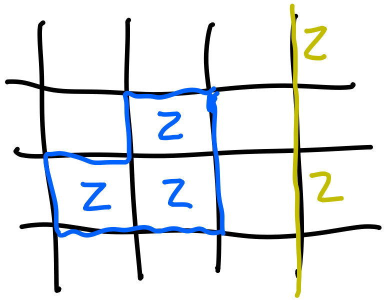
$\Rightarrow$ order of code distance $d \sim L$ in perturbation theory needed to lift degeneracy!

and: <u>Def:</u> <u>check soundness</u>: every stabilizer acting on M qubits is a product of $\lesssim M^2$ terms in H_0:

TC is a <u>topologically ordered</u> phase of matter:

- low-energy degeneracy depends on topology of surface
- no <u>local</u> order parameter distinguishes g.s.

Indeed you need a logical string operator to tell apart ground states!



→ stabilizers are closed loops / products of smaller loops

→ nonlocal logical operator winds around torus

Contrast w/ ordinary phases of matter defined by classical codes...

Thm: for repetition code $H_0 = -\sum_{i\sim j} Z_i Z_j$,

$H = H_0 + hV$ is a ferromagnet if h small & V is locally symmetric: $[V, X_1 \cdots X_n] = 0$ (every term commutes)

As a simple example we can consider the quantum Ising model:

$$H = -\sum_{i=1}^{n-1} Z_i Z_{i+1} - \sum_{i=1}^{n} (h X_i + \varepsilon Z_i)$$

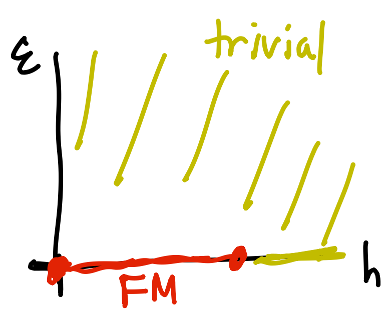
$\varepsilon = 0$ & h small $\Longrightarrow$ ferromagnet (in fact this model is solvable at $\varepsilon = 0$)

$\varepsilon \neq 0 \Longrightarrow$ unique ground state as $n \to \infty$

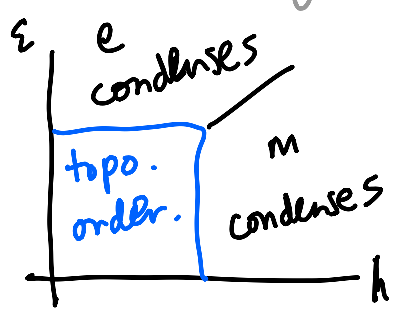
↳ perturbation $\varepsilon \sum Z_i$ violates Knill-Laflamme:

$\langle 00\cdots| \varepsilon \sum Z_i | 00\cdots\rangle = \varepsilon n$ $\quad$ $\langle 11\cdots| \varepsilon \sum Z_i | 11\cdots\rangle = -\varepsilon n$

Absolute stability of topological order vs. ferromagnet illustrated here:



$H_0' = -\sum ZZ - h\sum X - \varepsilon \sum Z$

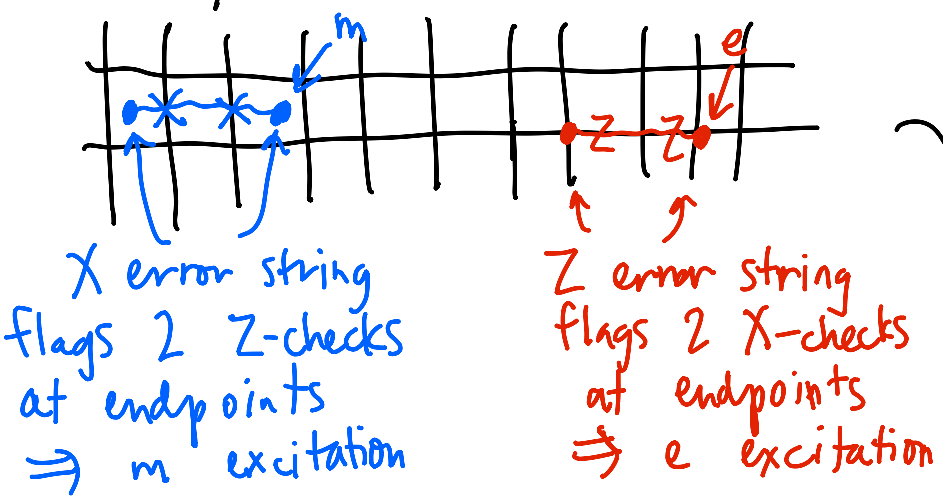


$H_0' = H_{TC} - h\sum X - \varepsilon \sum Z$

Key point is that earlier theorem guaranteed stability up to finite perturbation strength. This is robustness of gauge theory as a phase of matter!

Unfortunately this robustness does NOT extend to finite T, making QEC complicated. Let's start to understand why...

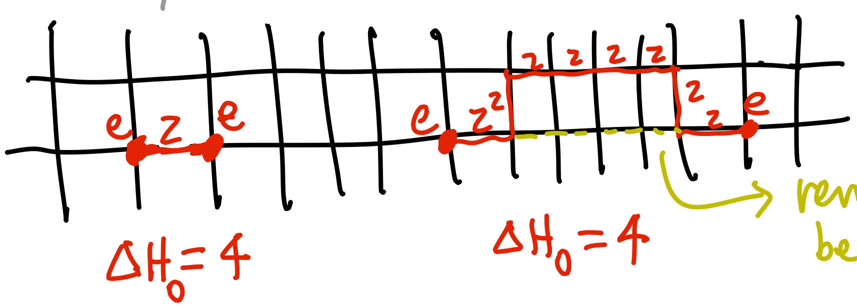
error syndrome in TC $\Rightarrow$ excitations (from cond-mat view)



e & m are created in pairs (much like domain walls in 1d Ising)

Def: <u>fractionalization</u>: isolated e & m can't be created, but isolated e & m can be independently moved.

The problem is that once we create e & m, no energy penalty to move them farther apart...



remember: there's NO difference between these two errors!

$$\Delta F_{e\ pair} \sim \Delta H_0 - T\Delta S_{e\ pair} = 4 - T\log\left[(L^2)^2\right] = 4(1 - T\log L)$$

$\Rightarrow$ for any $T > 0$, e & m <u>proliferate</u> due to finite energy barrier

This turns out to destroy topological order at any finite T.
We'll need to use an active decoder! (Discussed in later lecture...)

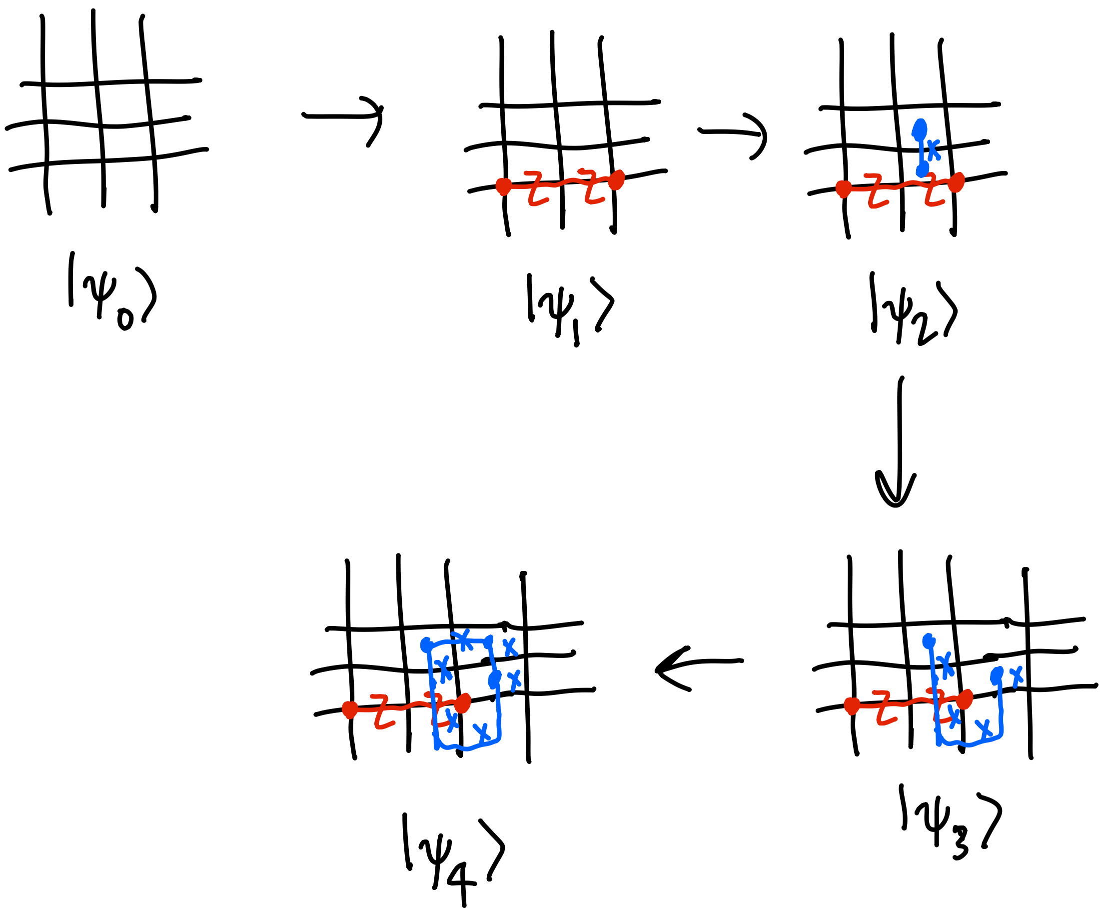
Recall: boson: $\psi(x_1, x_2) = \psi(x_2, x_1)$

fermion: $\psi(x_1, x_2) = -\psi(x_2, x_1)$

e & m are neither bosons or fermions!

e & m have non-trivial <u>braiding statistics</u> $\Rightarrow$ <u>Abelian anyons</u>

Consider the following thought experiment:



$$|\psi_4\rangle = (X\text{-loop})(Z\text{-string})|\psi_0\rangle = -(Z\text{-string})(X\text{-loop})|\psi_0\rangle$$
$$= -(Z\text{-string})|\psi_0\rangle = -|\psi_1\rangle$$

since X-loop is a stabilizer of the TC ground state.

$\Rightarrow$ <u>braiding</u> m around e causes phase of -1

Similar thought experiments show, because all X Paulis commute w/ each other e.g.

braiding e around e cause phase $+1$
m around m