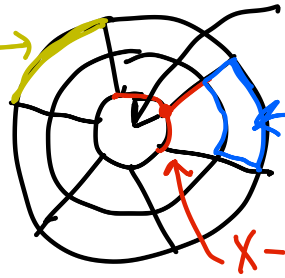
Toric code: logicals require non-trivial topology
↳ hard (but doable) to make torus in a lab!

Today: fix toric code to work on 2d planar grid by fixing the boundary conditions. leads us to surface code.

Consider toric code checks on cellulation of annulus:



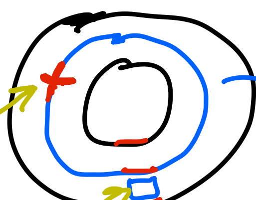
Z-check around face: $Z^{\partial f} = \prod_{e \in \partial f} Z_e$

X-check on vertex: $X^{dv} = \prod_{e \in dv} X_e$

$[X^{dv}, Z^{\partial f}] = 0$ by same logic as for toric code.

Prop: this code has $k=1$

Proof: $H^1(\text{annulus}; \mathbb{F}_2) \cong \mathbb{F}_2$: non-contractible cycle: ← famous fact from topology, here stated w/o proof.



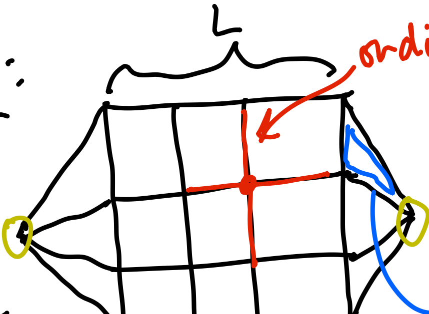
can't be shrunk to trivial loop ⇒ Z-logical

non-trivial cocycle between inner/outer boundary ⇒ X-logical

This is already a big improvement: annulus can be fit into plane.

But a cleaner/"more compact" thing is to use boundary conditions

Idea:



treat as "ghost vertex":
- remove X-check from stabilizer $\mathcal{S}$
- this removes two weight-L checks

keep Z-checks next to ghost vertex!

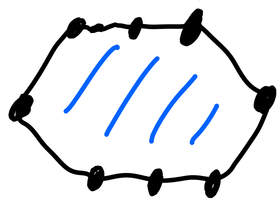
Result = surface code

Prop: surface also encodes $k=1$ logical qubit

Proof: We start w/ the following nice fact:

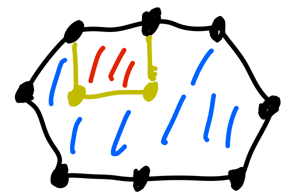
Lemma: for triangulation of a disk: $V-E+F=1$

Proof: start by adding in the outer boundary...



$$V-\underset{=V}{E}+\underset{=1}{F}=1$$

$\Rightarrow$ then add in internal faces one at a time

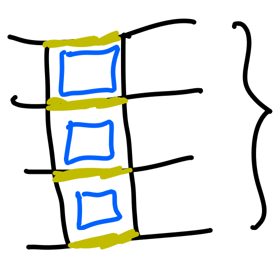


$F \to F + 1$
$E \to E + \Delta E$
$V \to V + (\Delta E - 1)$

$\Rightarrow V-E+F=1$ unchanged ∎

Lemma: $\#(X\text{-checks}) = V-2 = \operatorname{rank}(H_X)$
$\#(Z\text{-checks}) = F = \operatorname{rank}(H_Z)$

Proof: first equalities obvious by our construction; need to show that all checks are independent.
Let's show for Z-checks; similar argument for X-checks.



$\}$ L rows

L horizontal qubits
only contained in $L-1$ Z-checks

$$\hookrightarrow H_Z = \begin{pmatrix} \underbrace{\begin{matrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{matrix}}_{\text{full rank}} & \cdots \end{pmatrix}$$

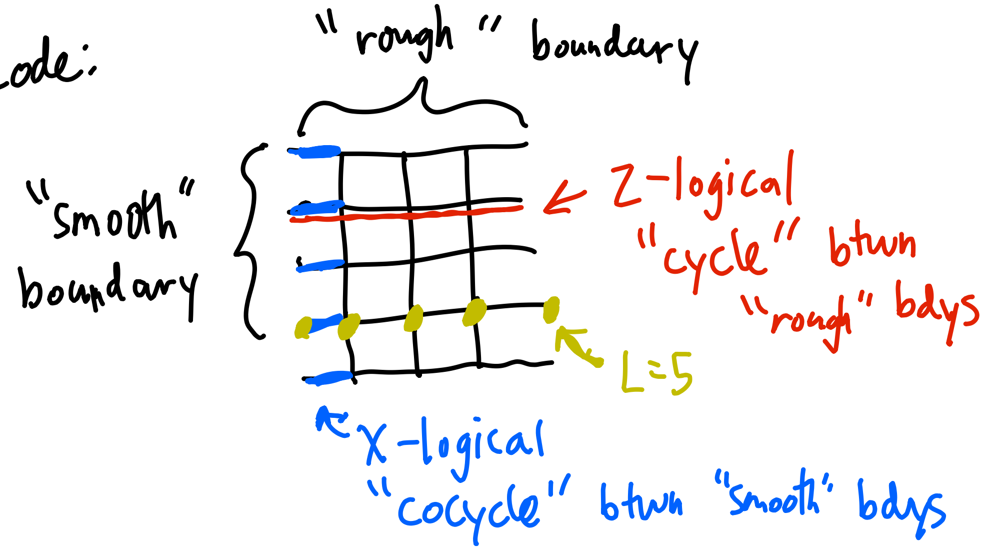
∎

Combine lemmas to finish proof:

$$k = n - \operatorname{rank}(H_X) - \operatorname{rank}(H_Z) = E-(V-2)-F = 2-(V-E+F) = 1.$$ ∎

So now we have a nice layout in a hole-free planar grid!
This is called a surface code!

Logicals in surface code:



count edges:

$$\underbrace{L(L-1)}_{\text{horizontal}} + \underbrace{(L-1)(L-2)}_{\text{vertical}} = n$$

count vertices:

$$L(L-2) = V$$

count faces:

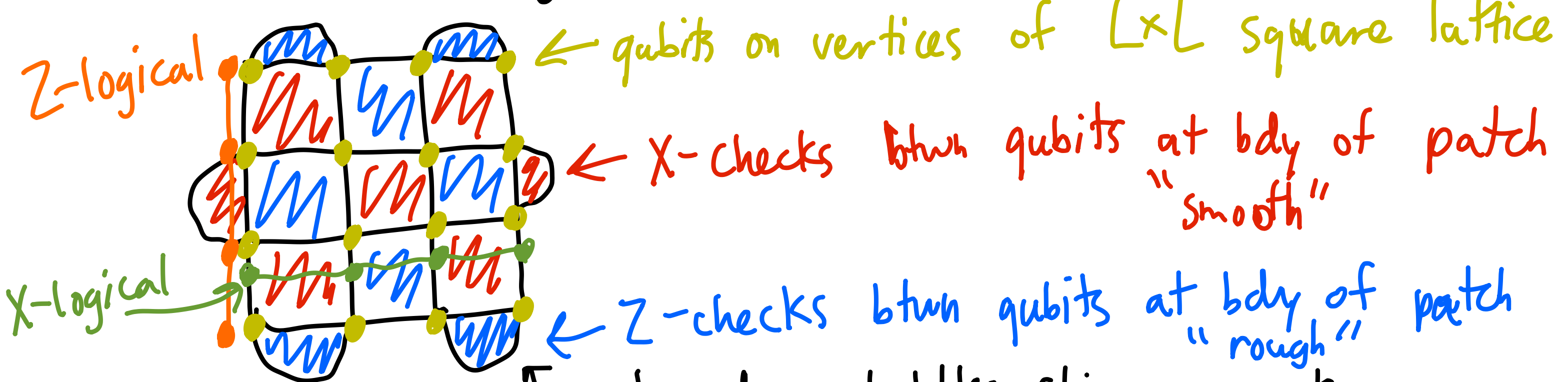
$$(L-1)^2 = F$$

$$\Rightarrow [[2(L-1)^2, 1, \underbrace{L-1}]]$$

$d_Z = L-1$ and $d_X = L$

We can make the surface code more efficient:

Goal: $[[L^2, 1, L]]$ surface code



We can be clever and spot by eye where representative logicals are.

Prop: $H_X H_Z^T = 0$

Proof: 2 faces share edge $\Rightarrow$ share 2 vertices.
(OK to have red X / blue Z touch)

2 faces share one vertex $\Rightarrow$ same color

This works b/c square lattice is bipartite.

Similar argument to before $\Rightarrow$ no redundant checks and $k=1$. ∎

In experiments people usually work w/ this improved surface code, So when we say "surface code" henceforth we refer to this.

So far we only have $k=1$ logical qubit in the surface code. Is it possible to have $k \gg 1$ (at large n) w/o sacrificing $d \sim L$?

<u>Thm:</u> (BPT bound): in D spatial dimensions w/ local checks:

$$k\, d^{\frac{2}{D-1}} \leq O(n).$$

Proof for $D=2$: rely on a useful:

<u>Cleaning lemma:</u> let R be a subset of qubits.

Every error on R correctable $\Rightarrow$ if $\underbrace{L \in N(\mathcal{S})/\mathcal{S}}_{\text{non-trivial logical}}$

$\exists$ $S \in \mathcal{S}$ s.t. LS has no support in R.

Proof: Let π_R restrict Pauli operators to region R

(Here imagine using the $\mathbb{F}_2$ formalism)

Define $W = \pi_R(\mathcal{S}) = \pi_R(\mathrm{im}(H^T))$

parity-check matrix.

Correctable error $\Rightarrow$ $W^\perp \subseteq \mathrm{im}(H^T)$

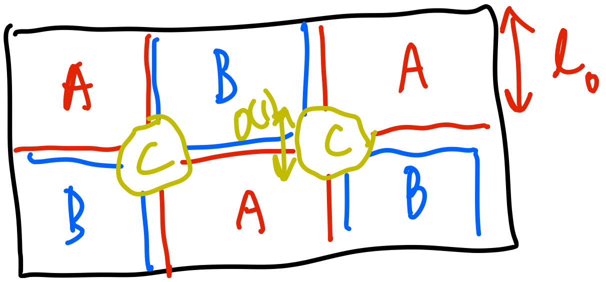
$\{x : w \cdot x = 0 \ \forall\, w \in W\}$ = Paulis that commute

If L is a logical, $q \cdot L = 0$ if $q \in W^\perp \Rightarrow L \in (W^\perp)^\perp = W$.

$\Rightarrow$ $\exists$ $w \in W$ w/ $\pi_R(w) = \pi_R(L)$.

Take logical $L' = L + w$ where $\pi_R(L') = 0$ has no support in R $\blacksquare$

Apply cleaning lemma to:



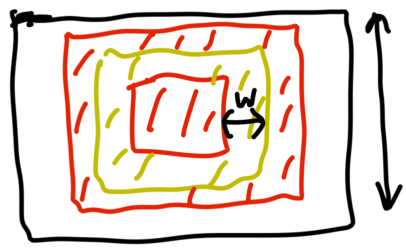
C is large enough to ensure no stabilizers connect different A & B patches

<u>Lemma:</u> If one patch of A cleanable, ALL of A cleanable. Ditto for B.

<u>Lemma:</u> Take $\ell_0 \sim d$ in 2D.

Proof: Apply 2 lemmas above to red/yellow regions in:

no stabilizer crosses 2 bdys →



final thickness $\sim d/4w$

$\blacksquare$

Clean all X-logicals out of A / all Z-logicals out of B.

for canonical pairing $L_X^\alpha \cdot L_Z^\beta = \delta^{\alpha\beta}$, $k \lesssim |C| \sim \left(\frac{\sqrt{n}}{\ell_0}\right)^2 \sim \frac{n}{d^2}$ $\blacksquare$

each logical pair must anticommute

This shows that the surface code is "best possible" if we demand d as large as possible, plus locality.

Now that we have a good 2d code, how can we protect information in it?

Surface code NOT passive memory at finite T.

$\Rightarrow$ need "active decoder" (subject of next lecture).

A very related, but <u>NOT</u> equivalent, observation is:

<u>Def:</u> $\rho_\beta := \frac{e^{-\beta H_0}}{\mathrm{tr}(e^{-\beta H_0})}$ is <u>Gibbs state</u>

where $H_0 = -\sum_{\text{checks } a} S_a$

<u>Prop:</u> partition function of surface code:

$$Z(\beta) = \mathrm{tr}(e^{-\beta H_0}) = 2(2\cosh\beta)^{n-1}$$

<u>Proof:</u> each X^{dv}, $Z^{\partial f}$ independent by previous proposition.

$\Rightarrow \exists$ 2 states in Hilbert space w/ <u>every syndrome</u> $\vec{s}$

$$\Rightarrow Z(\beta) = \sum_{|\vec{s}\rangle \otimes |\ell\rangle} \langle \vec{s}| e^{-\beta H_0} |\vec{s}\rangle = 2 \sum_{s_i \in \pm 1} e^{\beta(s_1 + s_2 + \cdots + s_{n-1})}$$

choose simultaneous eigenstates of each check

logical qubit

$$= 2(e^{\beta} + e^{-\beta})^{n-1} \quad \blacksquare$$

This implies no thermodynamic phase transition detectable via free energy. Suggests no robustness @ finite T?

This is consistent w/ picture at the end of last lecture: O(1) energy barrier to nucleate e & m and pull them apart. So at any β, eventually just see e & m diffusing $\Rightarrow$ large error strings.