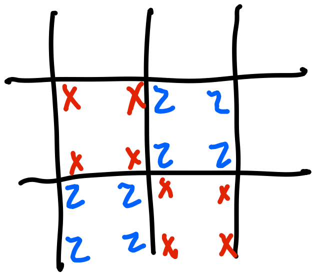
(rotated) surface code:



Goal: <u>quantum memory</u>:
protect logical qubit via syndrome measurement & feedforward

Begin by studying one round of syndrome extraction:

simple model: <u>independent</u> error on each qubit:

$$\text{probability} = \begin{cases} (1-p_e)^2 & I \\ p_e(1-p_e) & X \\ p_e(1-p_e) & Z \\ p_e^2 & Y \end{cases}$$

X & Z errors treated independently

X-type error Z-type error

$\mathbb{F}_2$ formalism: e_X & e_Z have $\mathbb{P}((e_{X/Z})_i = 1) = p_e$.

QEC measures <u>syndromes</u>: $s_X = H_X e_Z \qquad s_Z = H_Z e_X$

imperfect syndrome measurement: experimentalist observes

$$u_X = s_X + t_X \qquad u_Z = s_Z + t_Z \qquad \text{in } \mathbb{F}_2$$

w/ $\mathbb{P}(\underbrace{(t_{X/Z})_\alpha}_{} = 1) = p_m$

each individual syndrome measured incorrectly w/ probability p_m

<u>Prop:</u> After error channel, $\mathbb{P}[\text{logical error}] = p_{\text{fail}} \sim p_0^{\,d} = p_0^{\,L}$
if $d \gg \log n$

$\mathbb{P}$ = probability

even w/o decoding, very unlikely to see dangerous error in a single round of random errors.

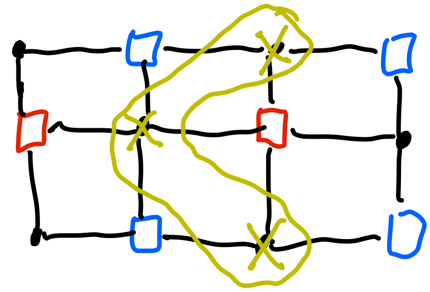
Proof: logical errors come from connected clusters w/ $\geq 50\%$ error density

Lemma: b/c code is qLDPC,

$\#\{$connected error clusters of size ℓ containing qubit $v\} := N_\ell \leq [e \, \Delta_B (\Delta_c - 1)]^{\ell-1}$

max # of X or Z checks containing qubit

max # of bits in check



connected error cluster

Proof sketch: at most $\Delta_B(\Delta_C-1)$ qubits share check w/ any one qubit.

Using generating functions, $N_\ell \leq$ # of trees of size ℓ, max degree $\Delta_B(\Delta_C-1)$ bounded as stated.

Using lemma,

$\mathbb{P}[$qubit v contained in X (or Z) error cluster of size $\ell] \leq p_e^\ell N_\ell$

$$\leq [e \Delta_B (\Delta_C - 1) p_e]^\ell$$

only 50% of qubits in cluster need error...

$p_{fail} \leq \mathbb{P}[$any error cluster of size $\ell] \leq 2 \cdot n \cdot [2 e \Delta_B (\Delta_C - 1) p_e]^\ell$

X or Z   # of qubits

$p_e < \frac{1}{4 e \Delta_B (\Delta_C - 1)} \Rightarrow$ if $\underbrace{\ell \sim d \gg \log n}$, $p_{fail} \leq [4 e \Delta_B (\Delta_C - 1)]^{d/2}$ ▨

logical error = connected cluster of size $\geq d$

quantum LDPC desirable: error clusters don't accumulate?

Goal: leverage this useful property to achieve:

Def: <u>fault-tolerance</u>: given finite $p_e, p_m > 0$ (or any realistic error model) $\lim_{n \to \infty} \mathbb{P}[\text{logical error}] = 0$.

Formal def subtle... but intuitively, can make logical error rates small.

Today talk about simplest goal:

Def: quantum memory: protect logical qubit for a long time:

$$\mathcal{R}_f\, \underbrace{(\tilde{\mathcal{R}}\mathcal{E}) \cdots (\tilde{\mathcal{R}}\mathcal{E})}_{T(n)\ \text{times}} (\rho_L) \simeq \rho_L$$

errors $(p_e > 0)$

noisy decoder $[p_m > 0]$

true / optimal?? decoder

Demand $\lim_{n\to\infty} T(n) = \infty$ so we can protect info for a long time at cost of a bigger code.

Given noisy syndromes:

$$u_X = H_X e_Z + t_X \qquad u_Z = H_Z e_X + t_Z$$

real errors, bad measurement

what is a good $\tilde{\mathcal{R}}$ to apply?

Key: CSS code $\Rightarrow$ decode X & Z errors separately!

discuss this one for convenience

Goal: find $$\mathbb{P}([e_X] \mid u_Z) = \sum_{\tilde{e} \in \operatorname{im} H_X^T} \frac{\mathbb{P}(u_Z \mid e_X + \tilde{e})\, \mathbb{P}(e_X + \tilde{e})}{\mathbb{P}(u_Z)}$$

equivalence class of errors: $\underbrace{e_X \sim e_X + H_X^T a}_{\text{same error up to stabilizers!}}$

Def: MAP decoder applies error e_X s.t. $\mathbb{P}([e_X] \mid u_Z)$ maximal.

This calculation is very messy in general! We'll want to find simpler algorithms...

Just like for classical 1d Ising, MAP decoder has interpretation as random Ising model...

$$\mathbb{P}(e_x) = \prod_{i=1}^{n} \begin{cases} p_e & (e_x)_i = 1 \\ 1-p_e & (e_x)_i = 0 \end{cases} \rightarrow \text{define Ising spin: } x_i = 1-2(e_x)_i \in \{\pm 1\}$$

$$= \exp\left(\sum_{i=1}^{n} h x_i\right) \quad \text{where} \quad e^{-2h} = \frac{p_e}{1-p_e}.$$

$\#$ of Z-checks $\to m_Z$

$$\mathbb{P}(u_Z|e_x) = \prod_{\alpha=1}^{m_Z} \begin{cases} p_m & (u_Z)_\alpha = 1+(H_Z e_x)_\alpha \\ 1-p_m & (u_Z)_\alpha = (H_Z e_x)_\alpha \end{cases} \quad \text{define: } s_\alpha = 1-2(u_Z)_\alpha$$

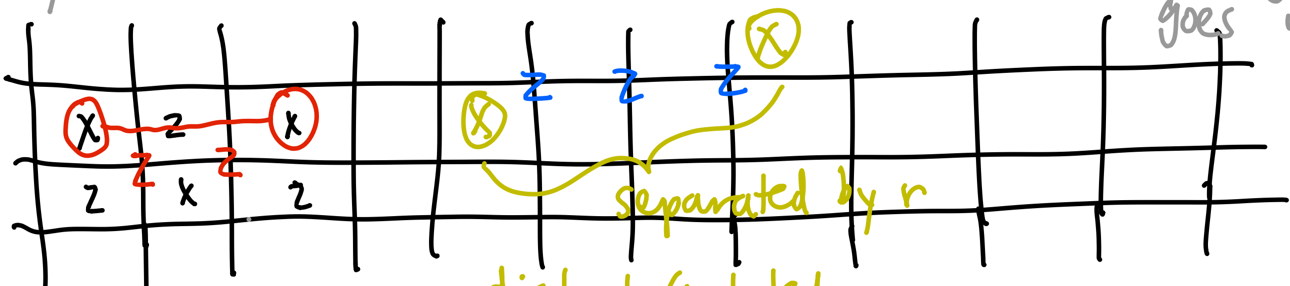
$$= \exp\Big(J \sum_{\alpha=1}^{m_Z} s_\alpha \prod_{\underbrace{i\sim\alpha}_{\text{qubit } i \text{ in Z-check } \alpha:\ (H_Z)_{\alpha i}=1}} x_i\Big)$$

MAP decoder searches for best equivalence class $\Rightarrow$ harder than finding ground state of disordered Ising model!

weaker decoder $\rightarrow$ just optimize over error e_x, not $[e_x]$?

$$\Rightarrow \text{ground state of} \quad H_{\text{easy},X} = -h\sum_{i=1}^{n} x_i - J\sum_{\alpha=1}^{m} s_\alpha \prod_{i\sim\alpha} x_i$$

This can still be complicated! The prop. above shows we're unlikely to introduce an error in surface code. But something still goes wrong...



distant/isolated flagged syndromes:

Small error flags nearby error string: detectable (w.h.p.)

$\mathbb{P}[\text{two bad syndrome measure}] \sim p_m^2$

vs. $\mathbb{P}[\text{error string of length } r] \sim [O(1)\cdot p_e]^r$

MAP thus generally ignores isolated syndromes. BUT

$\exists$ error cluster of size $\sim \log n$

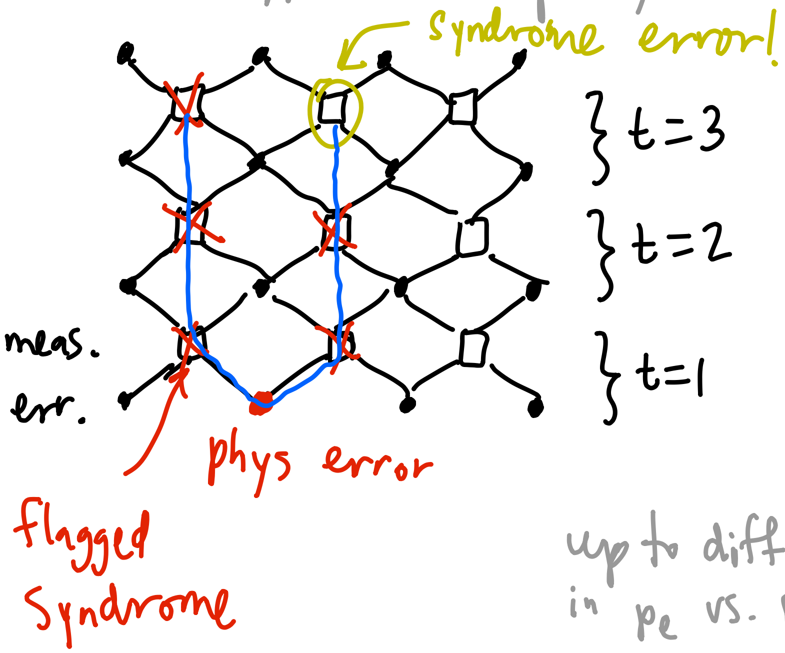
↳ this specific cluster more likely faulty syndrome meas.

↳ MAP ignores.

Process repeats $\Rightarrow$ logical error!

Fix this by repeating syndrome extraction (+noise) many times!

$\Rightarrow$ memory! For simplicity let's look at just one row of surf. code...



Space time loop connects phys error to syndromes!

add "as few" errors to close loop!

$\Rightarrow$ MWPM decoder!

up to diff. in p_e vs. p_m

The key is that we're now keeping track of errors in space & time:

$e_{x,t}$ = physical qubit errors (accumulated) at time t

$t_{z,t}$ = meas. fault at time t

$\Rightarrow$ measured syndrome: $u_{z,t} = t_{z,t} + H_z e_{x,t}$

The MAP Hamiltonian from above generalizes neatly:

$$H_{easy,x} = -\sum_t \Big[h \sum_{i=1}^{n} \underbrace{x_{i,t} x_{i,t-1}}_{\text{did new error take place?}} + \underbrace{J \sum_{\alpha=1}^{m_z} r_{\alpha,t} \prod_{i \sim \alpha} x_{i,t}}_{\text{was syndrome measured incorrectly}} \Big]$$

Thm: decode for $T \lesssim \text{poly}(d)\, e^{O(d)}$ ← distance $d \sim \sqrt{n}$ $\Rightarrow \mathbb{P}[\text{logical error}] \sim e^{-O(d)}$

Proof sketch: "Peierls argument" $\Rightarrow$ large loop of errors leading to logical is exponentially unlikely...

Outcome of decoding $\Rightarrow$ surface code w/ correctable error clust. $\Rightarrow$ info protected and we have quantum memory.

Def: below-threshold region: (p_e, p_m) for which $\lim_{n\to\infty} \mathbb{P}[\text{log. error}] = 0$.

The 2d surface code has one of the largest thresholds for any practical/scalable quantum code:

Thresholds for 2d surface codes:

no measurement errors: $p_e \approx 0.109$

w/ meas. errors & MAP decoder: $p_e = p_m \approx 0.031$

" " & MWPM decoder: $p_e = p_m \approx 0.029$

Experiments often quote lower numbers b/c they have a more involved noise model that accounts for syndrome extraction. We'll also want to understand how this can complicate QEC...

Thm: MAP decoding $\Rightarrow$ quantum memory if $d \gg \log n$ works in other qLDPC codes

Combined w/ ease of syndrome extraction (as we'll see later) this makes qLDPC codes main candidate for realizing fault-tolerant computation in experiment today.