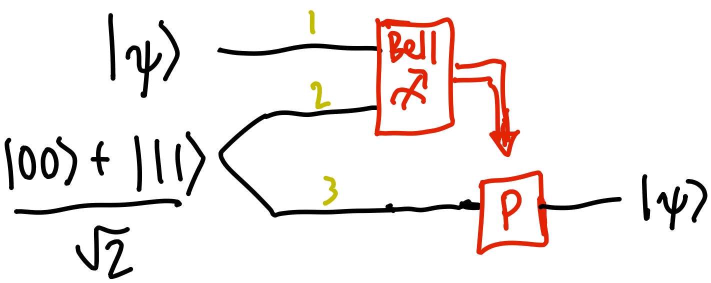
universal quantum computation needs non-Clifford gate

"magic" state

Today we will <u>sketch</u> how to do this in a fault-tolerant way, focusing less on optimizations of the procedure and more on big picture...

<u>Algorithm</u>: <u>quantum teleportation</u>: consider following circuit



Let's see why it works using the G-K stabilizer tracking formalism:

$$S = \begin{array}{cc} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} & \begin{array}{c} X_1 \\ X_2 \\ X_3 \\ Z_1 \\ Z_2 \\ Z_3 \end{array} \end{array} \xrightarrow[X_1X_2,\, Z_1Z_2]{\text{measure}} S' = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

stabilizers

w/ $\eta_{XX}, \eta_{ZZ} \in \{\pm 1\}$ measurement outcomes

$$L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow \text{stab. equivalent to } \tilde{L} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \xrightarrow{\text{post measurement}} \tilde{L}' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

logical ops for ψ

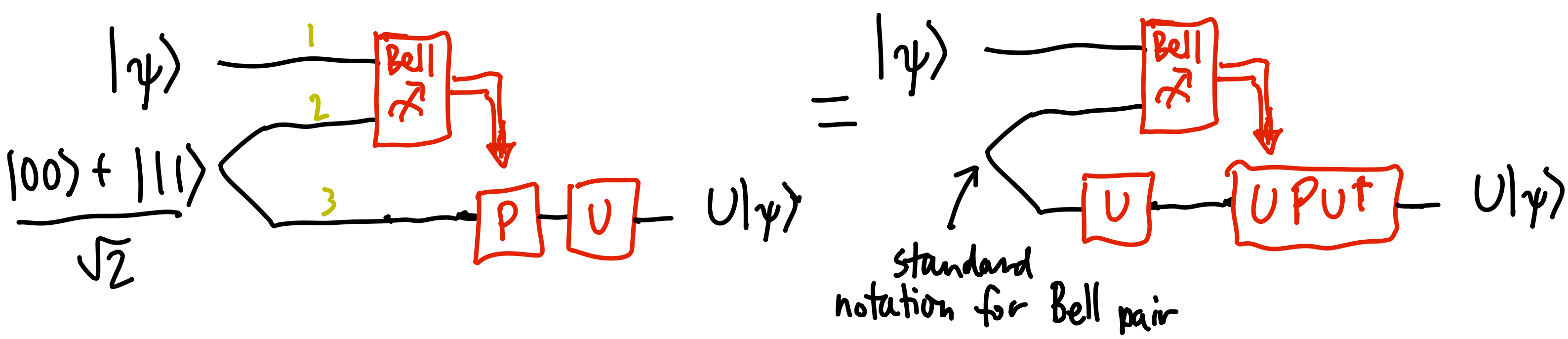
so qubit 3 must carry data from $|\psi\rangle$

Keep track of sign: $X_L = \eta_{XX} X_3$, $Z_L = \eta_{ZZ} Z_3$

Apply Pauli feedback:

(η_{XX}, η_{ZZ})	$(+1,+1)$	$(+1,-1)$	$(-1,+1)$	$(-1,-1)$
P	I	X	Z	Y

Algorithm: gate teleportation



Seems like the same thing, BUT given Clifford hierarchy...

If $U = T \in \mathcal{C}_1^{(3)}$ then $UPU^\dagger \in \mathcal{C}_1^{(2)}$ is Clifford

Goal: teleport on logical qubits encoded in fault-tolerant code w/ fault-tolerant (transversal?) Clifford gates.

Problem: original $\boxed{T}$ is noisy

$$\Rightarrow \mathbb{P}[\text{fail}] \sim \underbrace{\mathbb{P}[T \text{ fail}]}_{\sim p_{phys}\,?} + \underbrace{\mathbb{P}[\text{memory fail}]}_{\sim e^{-\alpha d} \to 0}$$

Key issue: how to get very high-fidelity T??

Def: magic state distillation takes many copies of noisy $|T\rangle_{L_1}$ encoded in code #1

$$|T\rangle = T|+\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + e^{i\pi/4}|1\rangle\right]$$

and outputs one higher quality $|T\rangle_{L_2}$ w/ probability < 1

encoded in code #2

Idea: use code to detect errors in $|T\rangle_{L_1}$

post-select on no detected errors

Def: if syndrome not perfect, discard + try again

To implement this procedure it's now time to formally define

Def: code concatenation: take 2 CSS codes

inner code C_{in} & outer code C_{out}

each "physical" qubit of (k_in copy of) C_{out} = logical qubit of C_{in}

Suppose $C_{in} = [[n_{in}, k_{in}, d_{in}]]$ and $C_{out} = [[n_{out}, k_{out}, d_{out}]]$

concatenated code is $[[n_{in} n_{out}, k_{in} k_{out}, d]]$ and has

$$H_X = \begin{pmatrix} I \otimes H_{X,in} \\ \underbrace{H_{X,out}}_{\text{parity-check}} \otimes \underbrace{G_{X,in}}_{\text{logical}} \end{pmatrix} \qquad G_X = \left(G_{X,out} \otimes G_{X,in} \right)$$

ditto for H_Z, G_Z

Example: Shor code:

$$H_{Z,in} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad H_{X,in} = 0 \qquad G_{X,in} = (1\ 1\ 1) \qquad G_{Z,in} = (1\ 0\ 0)$$

$$H_{Z,out} = 0 \qquad H_{X,out} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad G_{X,out} = (1\ 0\ 0) \qquad G_{Z,out} = (1\ 1\ 1)$$

$$\Rightarrow H_X = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \qquad H_Z = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$G_X = (1\ 1\ 1\ \ 0\ 0\ 0\ \ 0\ 0\ 0)$$
$$G_Z = (1\ 0\ 0\ \ 1\ 0\ 0\ \ 1\ 0\ 0)$$

Choose $C_{out} = [[15, 1, 3]]$ Reed-Muller code w/ transversal T.

$C_{in} = [[L^2, 1, L]]$ surface code ? or other fav CSS code

$$\underbrace{T_{15L^2}}_{\text{true logical } T} = \Big(T^{\dagger}_{L^2}\Big)^{\otimes 15}$$

$T^{\dagger}_{L^2}$ ← logical T on smaller patch

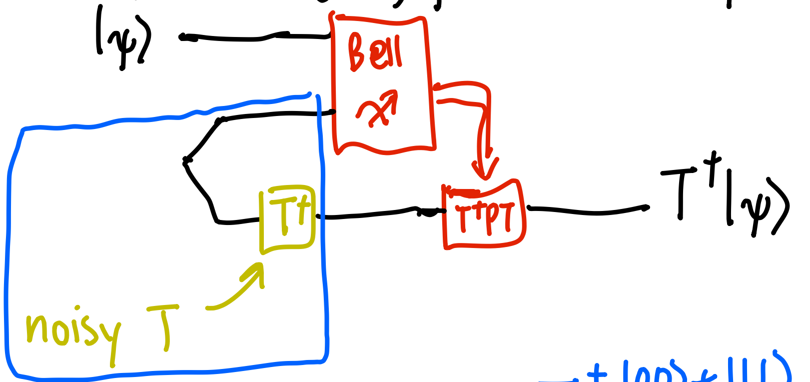
already looking transversal...

But we don't have a transversal T_{L^2}, so now what?

Implement T_{L^2} via gate teleportation!

NB: we're really in mixed state as seen by 1 qubit but procedure stays same

one (logical) qubit from 15 qubit outer code



repeat 15 times to apply logical T to (teleported) $15L^2$ code

in practice: $T^\dagger \frac{|00\rangle + |11\rangle}{\sqrt{2}} = CNOT_{2\to 1}\, T_2^\dagger |0\rangle \otimes |+\rangle$

noisy $|T^\dagger\rangle$ state

So now we have the noisy $|T\rangle$ encoded in a high distance code...

$$\mathbb{P}[H_{X,out} \otimes G_{X,in} \text{ syndrome measure fail}] \sim e^{-\alpha d} =: p_{circ,L}$$

$$\mathbb{P}[T_{15L^2}|+\rangle_{15L^2} \text{ failed}] \sim O(p_{circ,L}) + \underbrace{\mathbb{P}[p_{phys} \text{ errors undetected}]}_{p_T}$$

since RM had distance 3:

<u>Prop</u>: Assume Z errors during preparation of $|T\rangle$. Then:

$$p_T = 35\, p_{phys}^3 + O(p_{phys}^4)$$

<u>Proof</u>: Which weight 3 errors undetected?

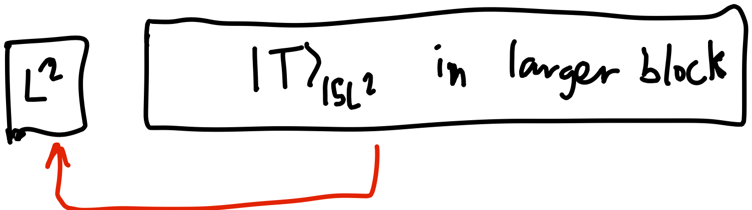
Recall: $H_X = \underbrace{\begin{pmatrix} \vdots & \vdots & \cdots \\ \vdots & 0 & \end{pmatrix}}_{15 \text{ nonzero vectors in } \mathbb{F}_2^4} \Rightarrow H_X \vec{e}_Z = 0$ (weight 3)

$\Leftrightarrow \underbrace{a + b = c}_{\text{errors occur in columns } a,b,c:} \in \mathbb{F}_2^4$

Number of choices of $(a,b,c) = \frac{1}{3!}(15\cdot 14) = 35$ $\blacksquare$

permute a/b/c $\nearrow$ $\nwarrow$ c fixed given a, b

It might be useful to push the distilled $|T\rangle$ back to a single L^2 code block, which can be done FT...



teleport $|T\rangle$ here? so stabilizer weight again smaller...

Follow the same state teleportation procedure...

$|+\rangle_{L^2} \otimes |T\rangle_{15L^2} \longrightarrow$ stabilized by X_{L,L^2}, logical stab of $15L^2$ code... smaller stabilizers...

Fault-tolerantly measure: $Z_{L,L^2} Z_{L,15L^2}$ involves 3 of 15 blocks (we'll discuss this step more next time...)

η_z

logical $X_{L,15L^2}$ destroyed but stab. equiv. $X_{L,L^2}X_{L,15L^2}$ preserved.

η_x

FT measure: $X_{L,15L^2} \Rightarrow$ logical data in $X_{L,L^2}\cdot\eta_x$, $Z_{L,L^2}\eta_z$

$\Rightarrow$ final state is $|T\rangle_{L^2} \otimes |\text{something}\rangle_{15L^2}$.

This procedure can be iterated to make higher quality magic $|T\rangle$ states in a process known as the magic state factory.

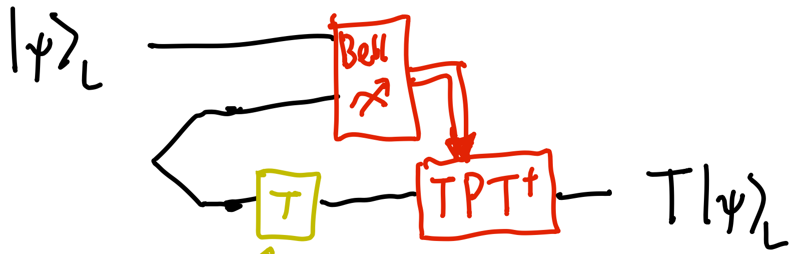
Magic state factory: n rounds of distillation $\Rightarrow$

$$p_n \approx 35\, p_{n-1}^3 + p_{\text{circ},n} \approx 35^{\frac{3^n-1}{3-1}} p_0^{3^n} < \left(\sqrt{35}\, p_0\right)^{3^n}$$

If $p_0 \approx 10^{-3}$, then $p_1 \approx 3.5\times 10^{-8}$, $p_2 \approx 1.5\times 10^{-21}$ etc.

Need $p_{circ,n} \lesssim 35\, p_{n-1}^3 \Rightarrow$ larger distance inner (surface) codes!

Finally, we can apply a FT T gate to encoded logical:



and we need to do this for every T!

from magic state factory (uses same T preparation strategy discussed before)

This painful procedure is a huge reason why so many estimates for the number of qubits needed for FT computation are so high.

Next time we discuss how this can all be done in a planar architecture.