As an alternative to expensive magic state distillation, we now turn to:

code switching: alternate route to universal gate set

Code A: transversal $\overline{T}$ (and not $\overline{H}$) $\rightleftarrows$ code B: transversal $\overline{H}$ (and not $\overline{T}$)

Need to choose A & B carefully so the switch is fault-tolerant...

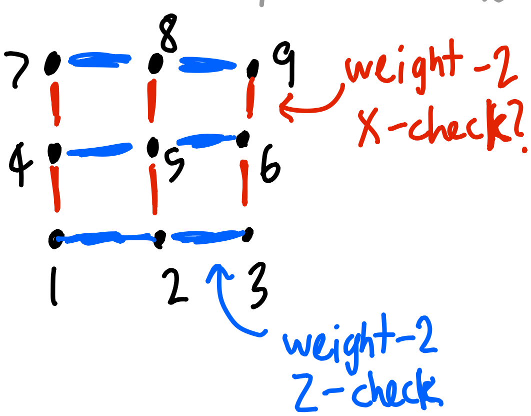
A natural solution will come from a possibly unexpected place...

Recall Shor code: Pauli stabilizer code w/

$$\mathcal{S} = \langle \underbrace{X_1X_2X_3X_4X_5X_6}_{\text{Problem: high-weight}}, X_4X_5X_6X_7X_8X_9, Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9 \rangle$$

(bad for circuit-level FT?)

Could we try to do QEC in alternate way here avoiding high weight stabs?



Consider

$$\mathcal{G} := \langle X_1X_4, X_4X_7, X_2X_5, X_5X_8, X_3X_6, X_6X_9, Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9 \rangle.$$

$\downarrow$

non-Abelian subgroup of Pauli group $\mathcal{P}_9$.

Def: *centralizer* of set H in group G:

$$C_G(H) := \{ g \in G : gh = hg \ \forall h \in H \}$$

$$C_{\mathcal{P}_9}(\mathcal{G}) = \langle \underbrace{X_1X_2X_3, X_4X_5X_6, X_7X_8X_9}_{\text{X-logicals of Shor code}}, \underbrace{Z_1Z_4Z_7, Z_2Z_5Z_8, Z_3Z_6Z_9}_{\text{Z-logicals!}} \rangle$$

$$C_{\mathcal{P}_9}(\mathcal{G}) \cap \mathcal{G} = \langle \underbrace{X_1X_2X_3X_4X_5X_6, X_4X_5X_6X_7X_8X_9}_{\text{big X-checks}}, \underbrace{Z_1Z_2Z_4Z_5Z_7Z_8, Z_2Z_3Z_5Z_6Z_8Z_9}_{\text{big Z-checks.}} \rangle$$

What if we'd taken $\mathcal{S} = C_{\mathcal{P}_9}(\mathcal{G}) \cap \mathcal{G}$?

↳ naive stabilizer code would have $k=5$ logical pairs:

$\boxed{(X_1X_2X_3, Z_1Z_4Z_7)}$; $\boxed{(X_1X_4, Z_1Z_2); (X_5X_8, Z_5Z_6)}$ etc...

↳ let's take this = real logical (representative)

↳ = "fake" gauge logical

⇒ do NOT protect info encoded here

→ (some) elements of $\mathcal{G} \setminus \mathcal{S}$

This is 9-qubit Bacon-Shor code.

- measure Z-generators $\mathcal{G}_Z$ (detects all X-error)
- measure X-generators $\mathcal{G}_X$
  - → mod $\mathcal{G}_X$, only equivalent X-errors are X_1, X_2, X_3
  - → previous $Z_1 \sim Z_2$ errors "distinguished" by random measurement outcomes of $\mathcal{G}_X$. ⇒ not meaningful.
- measure $\mathcal{G}_Z$ ⇒ only meaningfully distinguish Z_1, Z_4, Z_7
- etc.

switching btwn X/Z sectors = changing gauge

Key: X-logical error ⇒ one error in each of 3 cols
Z-logical error ⇒ " " rows

} 1-qubit error still detectable!

gauge-equiv $\binom{X}{Z}$ errors in same $\binom{\text{col}}{\text{row}}$

Dynamics in $C(\mathcal{S}) \cong (\mathbb{C}^2)^{\otimes 5}$:

$$|\psi_L\rangle \otimes |\psi_{G,1}\rangle \longrightarrow |\psi_L\rangle \otimes |\psi_{G,2}\rangle \rightarrow |\psi_L\rangle \otimes |\psi_{G,3}\rangle \rightarrow \cdots$$

logical gauge

gauge qubits NOT protected.

To understand why this can work at all, useful to now extend our formalism of Pauli stabilizer codes to subsystem stabilizer codes, of which the Bacon-Shor code is a simple example:

Def: <u>center</u> of a group $Z(G) := C_G(G)$ ← set of elements commuting w/ everything else.

Def: <u>Pauli subsystem stabilizer code</u> characterized by

$$\underbrace{Z(G) = \mathcal{S}}_{\text{stabilizer}} \leq \underbrace{\mathcal{G}}_{\text{gauge group}} \leq \underbrace{\mathcal{P}_n}_{\text{Pauli group on } n \text{ qubits}}$$

If $\Phi_{ph} = \{\pm 1, \pm i\}$ then: set of logical operators is

$$\mathcal{L} = C(\mathcal{S}) \setminus \Phi_{ph} \mathcal{G}$$ ← operations that aren't gauge and commute w/ stabilizers

Code distance $d := \min_{P \in \mathcal{L}} wt(P)$

Code is $[[n, k, r, d]]$ when:

$$|\mathcal{S}| = 4 \cdot 2^{n-k-r} \qquad |\mathcal{G}| = 4 \cdot 2^{n-k+r}$$

effective logical group is $\mathcal{P}_{logical} = \dfrac{C(\mathcal{S})}{\Phi_{ph}\mathcal{G}}$ (changing logical by gauge is fine)

bare logical group $\mathcal{P}_{logical, bare} = \dfrac{C(\mathcal{G})}{\Phi_{ph}\mathcal{S}}$ doesn't account for possibility a smaller logical may act alongside gauge transformation.

We state w/o proof the following useful fact

<u>Prop:</u> $\exists$ Clifford transform such that

$$\mathcal{G} = \langle \underbrace{X_1, Z_1, \ldots, X_r, Z_r}_{\text{gauge DoF}}, \underbrace{Z_{r+1}, \ldots, Z_{n-k}}_{\text{stabilizers}} \rangle$$

In this basis logicals are $\langle X_{n-k+1}, Z_{n-k+1}, \ldots, Z_n \rangle$.

Above proposition also makes transparent that measuring $\mathcal{G}$ can't corrupt logical info, even if state is not invariant

<u>Example:</u> Bacon-Shor is $[[9,1,4,3]]$ subsystem code

$d=3$ b/c as we saw, could correct up to gauge any single-qubit error.

We'll usually be interested in subsystem CSS codes. Let's review how they look.

If $\mathcal{G}$ is generated by $\langle \mathcal{G}_X, \mathcal{G}_Z \rangle$
(X-Pauli) (Z-Pauli)

"parity-check" matrices M_X and M_Z

Similar to before, introduce classical codes

$C_X^\perp = \mathrm{im}(M_X^T) \quad C_X = \ker(M_X) \quad C_Z^\perp = \mathrm{im}(M_Z^T) \quad C_Z = \ker(M_Z)$

stabilizer subspaces: $\mathrm{im}(H_X^T) := C_X^\perp \cap C_Z \qquad \mathrm{im}(H_Z^T) = C_Z^\perp \cap C_X$

logical spaces $\quad L_X := \dfrac{\ker(H_Z)}{\mathrm{im}(M_X^T)} \qquad L_Z := \dfrac{\ker(H_X)}{\mathrm{im}(M_Z^T)}$

mods out by both stabilizers and gauge

<u>Def:</u> <u>code switching</u> is a (fault-tolerant) procedure for switching btwn two codes while protecting logical info

<u>Def:</u> <u>gauge fixing</u> is a nice example where

$$\mathcal{S} = Z(\mathcal{G}) \subseteq \mathcal{S}_A, \mathcal{S}_B \subseteq \mathcal{G} \text{ w/ common logical representatives}$$

preserved by $\mathcal{S}$

code A: $|\bar{\psi}\rangle \otimes |g_A\rangle$
logical gauge

code B: $|\bar{\psi}\rangle \otimes |g_B\rangle$ ← different gauge

- switch btwn codes by measuring extra stabilizers of $\mathcal{S}_A$ vs. $\mathcal{S}_B$

- apply QEC / frame tracking to correct errors.

Here's an example where we can choose the A & B codes to have a transversal $\bar{T}$ and $\bar{H}$ coming from gauge-fixing.

Take: [[15, 7, 3]] quantum Hamming code:

$$H_X = H_Z = H_1 = \underbrace{\begin{pmatrix} 1 & 1 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ 1 & 1 & & 0 \\ 1 & 0 & & 1 \end{pmatrix}}_{\text{all 15 non-zero vectors in } \mathbb{F}_2^4}$$

versus [[15, 1, 3]] Reed-Muller code:

$$H_X = H_1, \quad H_Z = \begin{pmatrix} H_1 \\ \hline H_2 \end{pmatrix} \qquad H_2 \in \mathbb{F}_2^{6\times 15}$$

Hamming has transversal $\bar{H}^{\otimes 7}$ (logical $\bar{H}$ on all logical qubits)

RM has transversal $\bar{T}$

code switch? $\Rightarrow$ interpret Hamming as [[15, 1, 6, 3]] subsystem CSS

$$M_X^{Ham} = M_Z^{Ham} = \begin{pmatrix} H_1 \\ \hline H_2 \end{pmatrix}$$

H_1: "H"

H_2: "M but not H"

$\Rightarrow$ Reed-Muller is gauge-fixed

so $H_X = H_1$, $H_Z = \begin{pmatrix} H_1 \\ \hline H_2 \end{pmatrix}$

Now let's see to get a transversal gate set...:

Start w/ $|\bar{\psi}\rangle \in \mathcal{C}_{RM}$ encoded in Reed-Muller codespace.

$\hookrightarrow \bar{T}|\bar{\psi}\rangle = (T^{\dagger})^{\otimes 15}|\bar{\psi}\rangle$ due to transversal T.

But interpreted as a gauge-fixed codeword of quantum Hamming:

$$H^{\otimes 15}|\bar{\psi}\rangle = H^{\otimes 15}\Big[\underbrace{|\bar{\psi}_{Ham}\rangle}_{\text{logical}} \otimes \underbrace{|\overline{000000}\rangle_{Ham}}_{\text{gauge}}\Big] = |H\bar{\psi}_{Ham}\rangle \otimes \underbrace{|\overline{++++++}\rangle_{Ham}}_{\text{need to gauge fix back to RM ?}}$$

Gauge fixing achieved by measuring $(H_2)_Z$ stabilizers, apply ($(H_1)_X$-stabilizer equivalent) correction based on outcome

Claim: these are $(H_2)_X$ gauge transforms! (Intuitively b/c quantum Hamming code had $G_X = G_Z$, and so does RM...)

$$\begin{tikzcd}
 & G = \langle H_{1X}, H_{1Z}, H_{2X}, H_{2Z} \rangle & \\
S_{RM} = \langle H_{1X}, H_{1Z}, H_{2Z} \rangle \arrow[ur, phantom, "\subset"] \arrow[loop left, "(T^{\dagger})^{\otimes 15}"] \arrow[rr, leftrightarrow, "H^{\otimes 15}"] & & S_B = \langle H_{1X}, H_{1Z}, H_{2X} \rangle \arrow[ul, phantom, "\subset"] \\
 & S = \langle H_{1X}, H_{2X} \rangle \arrow[ul, phantom, "\subset"] \arrow[ur, phantom, "\subset"] &
\end{tikzcd}$$

logical gates

Reed-Muller code

15-qubit quantum Hamming stabilizers

Subsystem code has $d=3 \Rightarrow$ no single fault in procedure causes logical error

However we haven't thought yet about circuit-level noise — is that a problem? To fix, we'll want to look for a family of topological codes w/ H & T transversal gates...
this leads us to study of 2d vs. 3d color codes...