Goal of QEC: $(\mathcal{D} \circ \mathcal{N} \circ \mathcal{E})(\rho) = \rho$

decode — noise/error — encode — logical qubit(s)

Useful simplification: if $\rho_L \in$ codespace (we'll be more precise soon!)

then $(\mathcal{R} \circ \mathcal{N})(\rho_L) = \rho_L$. $\quad (\rho_L = \mathcal{E}(\rho))$

$\mathcal{R}$: recovery (within codespace)

Today we'll study the simplest possible quantum code that can detect & correct a single qubit X & Z type error!

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repetition code for X-errors:

3 qubits ; Hilbert space $\mathcal{H} = (\mathbb{C}^2)^{\otimes 3} = \text{span}\{|000\rangle, |001\rangle, \dots, |111\rangle\}$

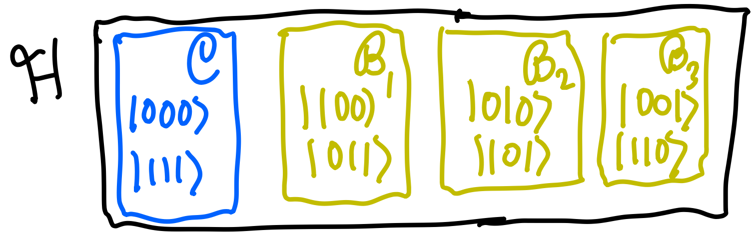
codespace $\mathcal{C} \subset \mathcal{H} = \text{span}\{|000\rangle, |111\rangle\}$

$|000\rangle \to |0_L\rangle$, $\quad |111\rangle \to |1_L\rangle$ (logical states)

This code can protect against a single Pauli X error:

$$X_1[\alpha|000\rangle + \beta|111\rangle] = \alpha|100\rangle + \beta|011\rangle \in \mathcal{B}_1$$
$$X_2[\alpha|000\rangle + \beta|111\rangle] = \alpha|010\rangle + \beta|101\rangle \in \mathcal{B}_2$$
$$X_3[\alpha|000\rangle + \beta|111\rangle] = \alpha|001\rangle + \beta|110\rangle \in \mathcal{B}_3$$

Notice that we can decompose $\mathcal{H}$ as:



Decoding strategy:

- measure sector $\mathcal{C}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$
- apply correction X_j if needed.

Define projectors:

$$P_{\mathcal{C}}|x_1x_2x_3\rangle = \begin{cases} |x_1x_2x_3\rangle & \text{if } x_1=x_2=x_3 \quad (\in\mathcal{C}) \\ 0 & \text{else} \quad (\notin\mathcal{C}) \end{cases}$$

$$R(\rho) = P_{\mathcal{C}}\,\rho\, P_{\mathcal{C}} + \sum_{j=1}^{3} X_j P_{\mathcal{B}_j}\,\rho\, P_{\mathcal{B}_j} X_j$$

defined analogously

This ought to be a valid quantum channel since it corresponds to a physical process, as discussed above. But let's check it!

$$R(\rho) = \sum_{j=0}^{3} K_j \rho K_j^\dagger, \quad K_0 = P_{\mathcal{C}}, \quad K_{j=1,2,3} = X_j P_{\mathcal{B}_j}$$

$$\sum_{j=0}^{3} K_j^\dagger K_j = P_{\mathcal{C}}^2 + \sum_{j=1}^{3} P_{\mathcal{B}_j} X_j^2 P_{\mathcal{B}_j} = P_{\mathcal{C}} + \sum_{j=1}^{3} P_{\mathcal{B}_j} = I$$

How would you do this in practice? Let's tease the general framework for quantum codes that we'll eventually introduce.

Define <u>stabilizers</u>: $S_1 = Z_1 Z_2$ & $S_2 = Z_2 Z_3$.

$S_1|000\rangle = (+1)^2|000\rangle = |000\rangle$ & $S_2|000\rangle = |000\rangle$.

$S_1|111\rangle = (-1)^2|111\rangle = |111\rangle$ & $S_2|111\rangle = |111\rangle$.

Checking the rest of the possibilities we find that:

$\boxed{\mathcal{C}:\ S_1 = 1,\ S_2 = 1}$ $\longrightarrow$ $P_{\mathcal{C}} = \frac{1+S_1}{2}\,\frac{1+S_2}{2}$

$\mathcal{B}_1:\ S_1 = -1,\ S_2 = 1$

$\mathcal{B}_2:\ S_1 = -1,\ S_2 = -1$

$\mathcal{B}_3:\ S_1 = 1,\ S_2 = -1$

Measuring stabilizers determines error!

<u>syndrome</u> = outcomes of $S_{1,2}$ measurements.

We saw by brute force that the α, β superposition was unchanged by moving between $\mathcal{C}$ & $\mathcal{B}_1$, e.g. There's a more elegant way to see this, which will be crucial later...

Define logical operators:

$$X_L = X_1 X_2 X_3 \qquad \& \qquad Z_L = Z_1$$

$$X_L\left[\alpha|000\rangle + \beta|111\rangle\right] = \beta|000\rangle + \alpha|111\rangle \quad (\text{swap } |0_L\rangle \text{ \& } |1_L\rangle)$$

$$Z_L\left[\alpha|000\rangle + \beta|111\rangle\right] = \alpha|000\rangle - \beta|111\rangle \quad (\text{phase on } |1_L\rangle)$$

do NOT measure X_L or Z_L $\Rightarrow$ destroys quantum logical!

Key: $[X_L, S_j] = [Z_L, S_j] = 0$. stabilizer commutes w/ logical!

obvious

check: $X_L S_1 = X_1 X_2 X_3 Z_1 Z_2 = XZ \otimes XZ \otimes X$

$$= (-ZX) \otimes (-ZX) \otimes X = S_1 X_L$$

Key fact: $XZ = -ZX$:

$$XZ|0\rangle = X|0\rangle = |1\rangle \qquad ZX|0\rangle = Z|1\rangle = -|1\rangle$$

$$XZ|1\rangle = -X|1\rangle = -|0\rangle \qquad ZX|1\rangle = Z|0\rangle = |0\rangle.$$

Errors need not commute w/ logical:

$$[X_1, Z_L] \neq 0.$$

so long as $[X_1, S_j] \neq 0$ for some j.

Prop: if detectable error E obeys: $ES_j = \eta_j S_j E$ $(\eta_j \in \{\pm 1\})$

Then syndrome measurement returns $S_j \to \eta_j$.

Proof: given codeword $|\psi\rangle \in \mathcal{C}$,

$$S_j E|\psi\rangle = \eta_j E S_j |\psi\rangle = \eta_j E|\psi\rangle$$

Since we're in eigenstate, S_j measurement deterministic ☒

Actually, linearity of QM makes this <u>very</u> powerful. We'll have to come back to this point later...

E.g. $X_1 S_1 = -S_1 X_1$ and $X_1 S_2 = S_2 X_1$

We can play a similar game to make a code that detects Z-errors (but not X).

Consider stabilizers: $S_1 = X_1 X_2$, $S_2 = X_2 X_3$.

$\Rightarrow$ logicals: $X_L = X_1$ $\qquad Z_L = Z_1 Z_2 Z_3$

Flipping the role of X & Z is fine b/c there is a unitary that does this!

Hadamard $H = \dfrac{X+Z}{\sqrt{2}}$: $\quad H^\dagger X H = Z$ & $H^\dagger Z H = X$.

So apply $H \otimes H \otimes H$: bit-flip rep. code $\rightarrow$ phase-flip rep. code.

$$\left.\begin{aligned} |0_L\rangle &= \tfrac{1}{2}\left[|000\rangle + |011\rangle + |101\rangle + |110\rangle\right] \\ |1_L\rangle &= \tfrac{1}{2}\left[|100\rangle + |010\rangle + |001\rangle + |111\rangle\right] \end{aligned}\right\}$$

Note: also applied logical Hadamard: $H^{\otimes 3}|000\rangle = \dfrac{|0_L\rangle + |1_L\rangle}{\sqrt{2}}$

Check: $S_1|0_L\rangle = \tfrac{1}{2}\left[|110\rangle + |101\rangle + |011\rangle + |000\rangle\right] = |0_L\rangle$

$X_L|0_L\rangle = \tfrac{1}{2}\left[|100\rangle + |111\rangle + |001\rangle + |010\rangle\right] = |1_L\rangle$ etc.

This time errors cause detectable phases in wave function!

$$Z_1|0_L\rangle = \tfrac{1}{2}\underbrace{\left[|000\rangle + |011\rangle - |101\rangle - |110\rangle\right]}_{\perp \text{ to } |0_L\rangle \text{ as half of phases now } -1}$$

Can also use commutators btwn Z-errors and stabilizers to detect...

Now let's see if we can combine these two codes to make a truly quantum code, protecting against X & Z error!

Code concatenation: take codeword(s)

$$|0_L\rangle = \frac{1}{2}\left[|0\rangle\otimes|0\rangle\otimes|0\rangle + |0\rangle\otimes|1\rangle\otimes|1\rangle + |1\rangle\otimes|0\rangle\otimes|1\rangle + |1\rangle\otimes|1\rangle\otimes|0\rangle\right]$$

replace w/ logical $|0\rangle$ of another code!

$|0\rangle \to |000\rangle$ $\qquad |1\rangle \to |111\rangle$

Result: 9-qubit Shor code:

$$|0_L\rangle = \frac{1}{2}\left[|\underset{123456789}{000000000}\rangle + |000111111\rangle + |111000111\rangle + |111111000\rangle\right]$$

$$|1_L\rangle = \frac{1}{2}\left[|111000000\rangle + |000111000\rangle + |000000111\rangle + |111111111\rangle\right]$$

This gets super ugly very fast! This is why it's so nice to talk about stabilizers because...

stabilizers: $S_1 = X_1X_2 \to X_{(L1)}X_{(L2)} = (X_1X_2X_3)(X_4X_5X_6)$

$S_2 \to X_4X_5X_6X_7X_8X_9$

AND stabilizers for (L1), (L2), (L3) codes:

$$Z_1Z_2,\ Z_2Z_3,\ Z_4Z_5,\ Z_5Z_6,\ Z_7Z_8,\ Z_8Z_9$$

Logicals transform:

$$X_L = X_{(L1)} = X_1X_2X_3$$

$$Z_L = Z_{(L1)}Z_{(L2)}Z_{(L3)} = Z_1Z_4Z_7$$

Claim: $[X_L, S_j] = [Z_L, S_j] = 0$ for each stabilizer S_j.

AND $\boxed{[S_j, S_{j'}] = 0.}$

↳ we can simultaneously measure ALL S_j to detect errors!

Strategy identical to before?!

X-error: measure $Z_1Z_2, \ldots, Z_8Z_9$.

$\Rightarrow$ detect ≤ 1 X-error in each block!

e.g. $|00|0000|0\rangle$

$Z_2Z_3 = -1$ $\quad$ $Z_7Z_8 = Z_8Z_9 = -1$

Z-error: measure S_1 & S_2?

but: $Z_j S_1 = -S_1 Z_j$ & $Z_j S_2 = S_2 Z_j$

for any $j = 1, 2, 3$?

OK! $\underbrace{Z_1 |\psi\rangle}_{\in C} = Z_1 (Z_1 Z_2) |\psi\rangle = Z_2 |\psi\rangle$

we can't tell apart errors that are "stabilizer-equivalent"

This is an extremely important point! We'll revisit again!

$\Rightarrow$ only errors to catch are Z_1, Z_4, Z_7!

Y-errors? Since $Y_j = -iZ_jX_j$, catch by detecting Z_j & X_j separately!