

Today:

code switching:

"2d" color code
(transversal H)

3d color code
(transversal T)

↔ actually we won't fully reduce to 2d for simplicity...

Def: 3-colex: cellulation of 3-sphere

- each volume cell assigned unique color: ● ● ● ●
- one cell of each color meet at vertex

color code $\Rightarrow$ qubits labeled by vertices v .

Before explaining how color code works we need to establish some useful facts about 3-colex...

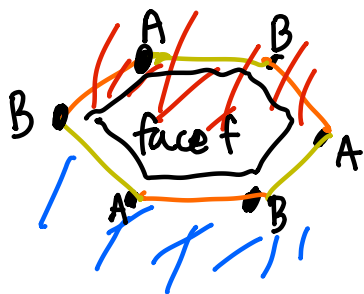


Prop: edge e touches 3 cells of different colors
 $\Rightarrow$ color edge e by missing color!

$\Rightarrow$ Prop: each vertex has one edge of each color.

$\Rightarrow$ Prop: every face has even # of vertices.

Proof: Suppose face f separates red/blue cells. then edges around face alternate btwn orange/yellow;



Two orange edges can't touch

$\Rightarrow$ even number of edges

$\Rightarrow$ even number of vertices



Cor: vertices form a bipartite graph: divide up vertices into sets A & B such that A vertex only adjacent to B vertices, vice versa

Given subset M of vertices, define $w(M) = |A \cap M| - |B \cap M|$
 $A = \overset{\uparrow}{\text{set of "A" vertices...}}$

Cor: for any edge e , face f , or 3-cell c :

$$w(e) = w(f) = w(c) = 0$$

Proof: First 2 follow immediately from previous corollary.

Suppose cell c bounded by faces $f_1, \dots, f_q$. Then

$$3 |A \cap c| = \sum_{j=1}^q |A \cap f_j| \quad (\text{each vertex contained in 3 faces at bdy})$$

$$\Rightarrow w(c) = \frac{1}{3} \sum_{j=1}^q [|A \cap f_j| - |B \cap f_j|] = \frac{1}{3} \sum_{j=1}^q w(f_j) = 0.$$

Def: 3d $(0,1)$ color code: define stabilizer group

$$\mathcal{S}_{(0,1)} := \langle X^c, Z^f \rangle$$

$$X^c := \prod_{v \in \text{cell } c} X_v \quad Z^f := \prod_{v \in \text{face } f} Z_v$$

Prop: for every c, f , $[X^c, Z^f] = 0$.

Proof: true if $c \cap f = \emptyset$.

If c & f share a vertex:



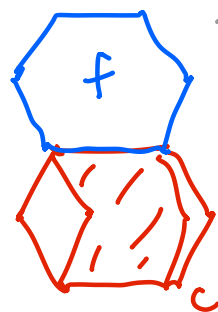
(f contained in c)

option 1:



$f \subset c$

option 2:



f & c
share an
edge;
 $|f \cap c| = 2$

Previous corollary $\Rightarrow |f \cap c| = \begin{cases} |f| & \text{option 1} \\ 2 & \text{option 2} \end{cases} = 0 \pmod{2}$

X & Z stab. share even num of qubits $\Rightarrow$ they commute. $\square$

A non-reduced representative of logical operators is:

Counting k (# of logicals) requires specifying boundary conditions.

one choice: take closed 3-sphere cellulation w/ 3-collex
 $\hookrightarrow$ even # of vertices

caveat:
not good
for code
distance,
but easier to
understand...

$\rightarrow$ remove one vertex & corresponding checks.

Thm: on this 3-collex, $k=1$. Logical Paulis are:

$$\bar{X} = \prod_v X_v \quad \bar{Z} = \prod_v Z_v$$

Proof: odd # of vertices so $\{\bar{X}, \bar{Z}\} = 0$. ($k \geq 1$)

To show $k=1$ exactly, need to evaluate $\text{rank}(H_X)$ & $\text{rank}(H_Z)$.

lemma: if we restore removed vertex, $k=0$.

Proof: Pick color a , define $j_a(e) := \begin{cases} x_u + x_v & (uv)=e \text{ color } a \\ 0 & e \text{ not color } a \end{cases}$

for a binary vector $x \in \mathbb{F}_2^V$ [denoting candidate X -logicals!]

every loop contractible and if $x \in \ker(H_Z)$,

$$\sum_{e \in \text{face } f} j_a(e) = 0 \text{ for every } f \quad \begin{array}{l} \text{[if } f \text{ has edges of color } a, \\ \text{follow from } x \in \ker(H_Z) \\ \text{trivial otherwise]} \end{array}$$

$$\Rightarrow j_a(e) = p_a(u) + p_a(v) \quad \text{[from homology/topology considerations]}$$

$$\Rightarrow x_v + \sum_{a=1}^4 p_a(v) = \text{const.} \quad \begin{array}{l} \text{similar to toric code...} \\ \leftarrow \text{follows from combining earlier eqs.} \end{array}$$

And since cell c has no edge of same color: $p_a(u)$ const. on cell of color a .

Up to global const. $x \in \text{im}(H_x^T)$.

$\hookrightarrow$ but for any color: $\prod_{\substack{v \in C \\ C \text{ has color } a}} X_v = \prod_{\text{all } v} X_v$

$\Rightarrow \ker(H_Z) = \text{im}(H_x^T) \Rightarrow k=0.$ $\square$

Now discuss what happens after removing a single vertex.

X -check redundancies $\Rightarrow \text{rank}(H_x) \rightarrow \text{rank}(H_x) - 1$

Do a similar thing for Z -checks by noticing:

lemma: if $\{f_1, \dots, f_r\}$ are cells separating cell of color a from cell of color $a' \neq a$:

$$\prod_{v \in C} Z_v = \prod_{j=1}^r \prod_{v \in f_j} Z_v$$

Proof follows from fact each vertex would show up in each set of faces for any a' exactly once.

$\Rightarrow$ 6 removed face checks (one for each color pair)

lemma $\Rightarrow$ all but 1 redundant. $\text{rank}(H_Z) \rightarrow \text{rank}(H_Z) - 1.$

So: $n \rightarrow n-1$, $\text{rank}(H_{x,z}) \rightarrow \text{rank}(H_{x,z}) - 1 \Rightarrow k \rightarrow k+1$ $\square$

Thm: 3d color code has transversal logical $\bar{T}$ gate.

If removed vertex from B , then

$$\bar{T} := \prod_{v \in A} T_v \prod_{v \in B} T_v^\dagger$$

$\nwarrow$ all 1 vector

Proof: codewords are: $|\bar{x}\rangle := \sum_{u \in \text{im}(H_x^T)} |u + \bar{x} \cdot 1\rangle$

$$\bar{T} |\bar{x}\rangle = \sum_{u \in \text{im}(H_x^T)} e^{i\pi/4 \cdot \omega(u + \bar{x} \cdot 1)} |u + \bar{x} \cdot 1\rangle$$

Note: $\omega(u + \bar{x} \cdot 1) = (1 - 2\bar{x})\omega(u) + \bar{x}\omega(1)$

$$w(\bar{x} 1) = (|A| - |B|) \bar{x} = \bar{x} \quad \text{b/c product of X-checks cancels vertices...}$$

$$w(u) = \sum_{\substack{\text{cell } c \in u \\ \uparrow \\ \text{X-checks generated by cell } c}} w(c) - 2 \sum_{c \leq c'} w(cnc') + 4 \sum_{c \leq c' \leq c''} w(cnc'nc'') - 8 \sum_{c \leq c' \leq c'' \leq c'''} w(cnc'nc''nc''')$$

cell: $w=0$
face or edge: $w=0$
edge: $w=0$

X-checks generated by cell c

~~$$- 8 \sum_{c \leq c' \leq c'' \leq c'''} w(cnc'nc''nc''')$$~~

ignore (mod 8)

so $w(u) = 0 \pmod{8}$. $\Rightarrow T|\bar{x}\rangle = e^{i\pi/8 \bar{x}} |\bar{x}\rangle$ is logical T ~~gate~~

Def: 3d (0,0) gauge color code:

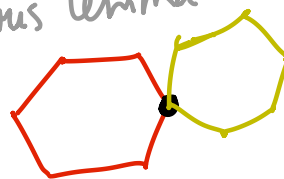
gauge group: $G_{(0,0)} := \langle X^f, Z^f \rangle$

both kinds of stab on faces

$$S_{(0,0)} := \langle X^c, Z^c \rangle$$

Note: $S_{(0,0)}$ justified by previous lemma

2 faces can meet at a single vertex:



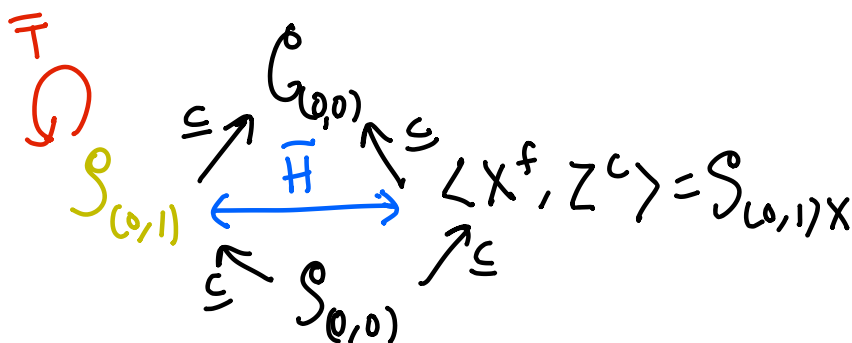
so: $S_{(0,0)} \subseteq S_{(0,1)} \subseteq G_{(0,0)} \Rightarrow$ code switching!

same 3-codex manifold $\Rightarrow \bar{X}, \bar{Z}$ same logical representatives.

By inspection we see that: (as before, commute w/ all face (gauge) operators)

Prop: (0,0) gauge code has transversal $\bar{H} = \prod_{\text{all } v} H_v$

So we code switch via same procedure as before



So in a nutshell strategy will be to switch b/w $S_{(0,0)}$ and $S_{(0,1)}$ codes.

After transversal $\bar{H}$ switch back to $S_{(0,1)}$ by measuring Z^f .

outcomes: $Z^f = \eta^f$ ↙

Apply suitable X^f gauge correction to return back to $S_{(0,1)}$...

What remains: is gauge fixing from $S_{(0,1)} \times \rightarrow S_{(0,1)}$
fault-tolerant?