

Recall 3d gauge color code on 3-cplex:



- qubits on vertices

- each vertex next to one cell of each color

gauge group: $G := \langle X^f, Z^f \rangle$ [live on faces]

where $X^f := \prod_{v \text{ around } f} X_v$ etc.

stabilizers $S := \langle X^c, Z^c \rangle$ [live on cells]

Recall: if cell c has color a , for any color $a \neq a'$:

$$X^c = \prod_{\substack{f \text{ adj. to } c \\ f \text{ b/w } a \text{ \& } a' \text{ cells}}} X^f$$

← there are 3 ways to check this! $\Rightarrow$ redundancy

Consider starting in X^f -stabilized state & measuring Z^f
 $Z^f \neq 1$ from:

① $[X^f, Z^{f'}] \neq 0$ ["gauge syndrome"]

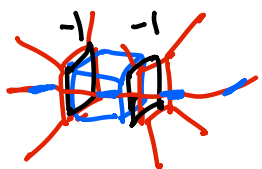
② physical errors

③ measurement errors

This is an LDPC code so we should formally expect fault-tolerance but we'll see we can actually find superior decoding vs. 2d TC.

① gauge syndrome = harmless but let's still understand what they look like

Since S contains Z^c , flipped syndromes Z^f form closed loops



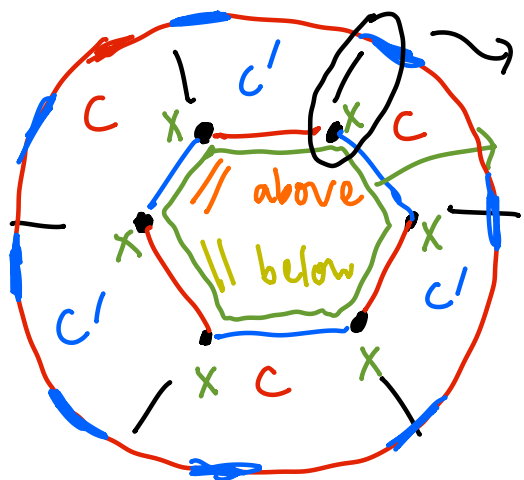
red/blue flux line closed.

← same for all 6 color pairs

Note that this data is independent for each of 6 sectors (for now!)

fault-tolerant T gate $\Rightarrow$ need to remove gauge syndrome.

Prop: red/blue loop of flux removed by orange/yellow X^f repair



this X anticommutes w/ face check repairs flux loop

Proof intuition:

- repair preserve S_Z
 $\Rightarrow$ must be in G_X [or logical error]

- repair on o/y face
 $\Rightarrow$ commute w/ any Z^f if f has o or y

Note: repair loop of length $L \Rightarrow$ need $\prod X^f$ acting on $\lesssim L^2$ faces
 (Analogous to check soundness from toric code: any loop of length L encloses area $\lesssim L^2$)

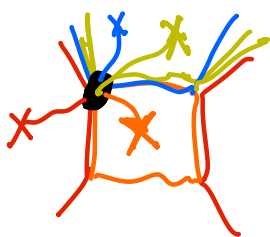
Now let's also include some X data errors...

② If X data error is gauge $\Rightarrow$ harmless, handled by ①

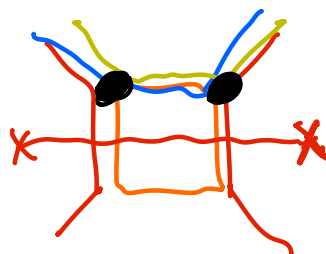
$\Rightarrow$ remaining errors are logical/uncorrectable or flag Z^c .

interpret: color changes inside cells.

Looking at slice through a 3d plane...



one error toggles change of each color



error on red edge
 Cancel all but red string of cells

And you can consider other configurations, Key point is that change is not only conserved in $\mathbb{F}_2$, but also btwn colored cells...

3d color code = LDPC. $\Rightarrow$

Def: given cluster C of connected vertices [on Tanner graph]

call C α -dense if $\geq \alpha|C|$ qubits $\in C$ have X errors.

Analogous to our earlier discussions about decoding we expect that.

Prop: MWPM decoder succeeds if no $1/2$ -dense cluster of size $\geq d$.

LDPC codes are useful precisely because: under independent error model:

Thm: if noise $p < p_c \leftarrow \text{some } O(1) \text{ constant}$

$$p_{\text{fail}} = \mathbb{P}[\exists \text{ } 1/2\text{-dense cluster of size } \geq d] \leq n \cdot e^{-cd}$$

Proof: Fix cluster C of size d .

$$\sum_{j=\lceil d/2 \rceil}^d \binom{d}{j} < 2^d \text{ ways to put } 1/2\text{-dense error inside}$$

$$\text{prob. of one such error} < p^{d/2}$$

$\leftarrow$ note: here referring to adjacency on qubit graph

Lemma: # of clusters C containing fixed qubit $\leq (ez)^d$

where $z \leq \Delta_B(\Delta_C - 1) = \# \text{ of qubits share check.}$

Proof omitted, based on combinatorics.

Combine: $p_{\text{fail}} < \sum_{s=d}^n n (ez)^s 2^s$

of places to root C # of C # of error patterns

$p^{n/2} \leftarrow \text{prob. of seeing error}$

$$p_{\text{fail}} < \frac{n}{1-2ez\sqrt{p}} (2ez\sqrt{p})^d < 1 \text{ if } p \text{ small. } p_c \geq \frac{1}{(2ez)^2}$$

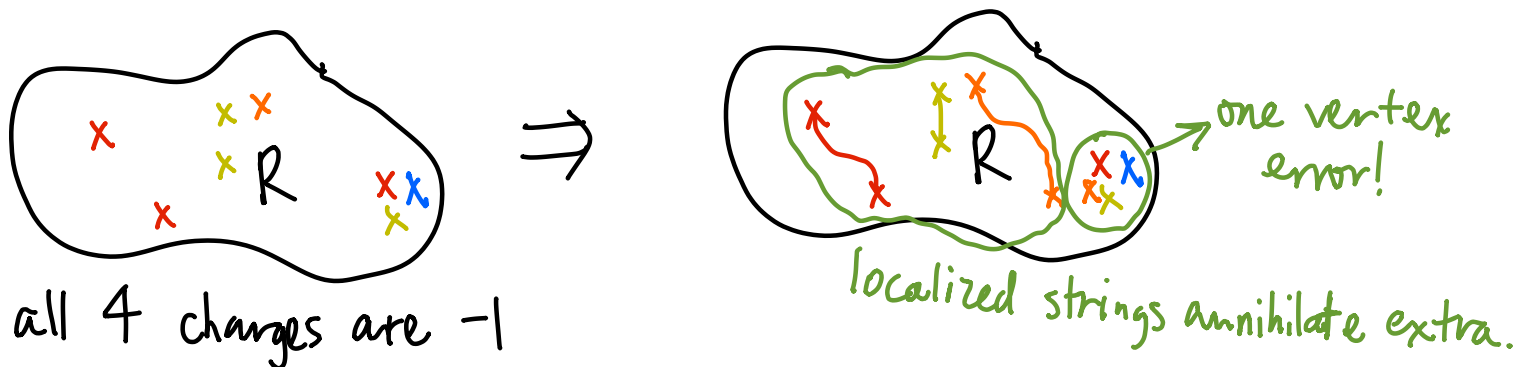
Now let's return to 3d color code. We focus on bulk properties for simplicity:

Prop: let R be a face-connected set of cells. If for each color a :

$$\prod_{\substack{c \text{ has color } a \\ c \in R}} Z^c = \eta^R \in \{\pm 1\} \quad (\text{namely all changes agree!})$$

then $\exists$ error E supported in R producing syndrome.

Proof sketch: Use the 2 rules we introduced before!



The connected-cell vertex error must be in R entirely, consistent w/ our assumptions.

Cor: repairing error E has weight $\leq K \cdot |R|$

$K = \# \text{ of vertices in a cell}$

Punchline: local repairs exist for charge-neutral clusters. But, same thing was also true for 2d TC... what's important here?

③ Now consider **faulty** Z^f measurements.

$$\Rightarrow \text{syndrome } f_Z = \underbrace{H_Z}_{\text{phys. + gauge error}} e_x + \underbrace{t_Z}_{\text{gauge checks}} \quad \text{in } \mathbb{F}_2$$

The constraints from consistency at each cell (closed flux lines) $\Rightarrow$

Def: Metacheck is a constraint on physical syndromes

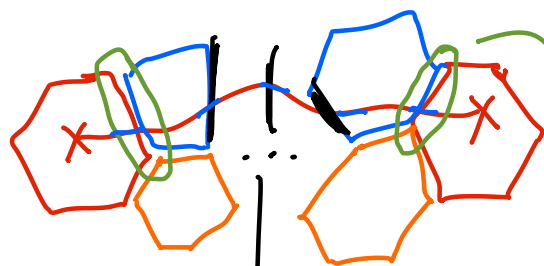
$$\Rightarrow \text{matrix } M_Z \text{ w/ } M_Z H_Z = 0$$

$$\Rightarrow M_Z f_Z = M_Z [H_Z e_X + t_Z] = \boxed{M_Z t_Z} \quad \text{metachecks help us detect faulty readout!}$$

Use MAP/minimum-weight decoding to guess most likely $\hat{t}_Z$:

$$M_Z t_Z = M_Z \hat{t}_Z \quad \text{w/ } |\hat{t}_Z| \text{ smallest.}$$

Now study corrected/charge-valid syndrome $\hat{f}_Z = f_Z + \hat{t}_Z$.



suppose these $-1 \Rightarrow$ faulty syndrome readout even in $\hat{f}_Z$

need $\frac{1}{2}$ -dense cluster of faulty Z^f readout: rb flux line must connect charges!

More precise: Prop: suppose $t_Z + \hat{t}_Z = t_Z^{\text{bad}}$; $M_Z t_Z^{\text{bad}} = 0$
↑ residual error

Then there were measurement faults in at least 50% of t_Z^{bad} , in every connected cluster.

Proof: let C = set of cells connected to face in cluster.

Gauge-equivalent syndrome: $t_Z \sim t_Z + 1_C =: t_Z^{\text{alt}}$

By assumption t_Z^{bad} reduced:

Restricting to cluster C :

$$(1_C)_f = \begin{cases} 1 & f \in \partial C \\ 0 & \text{else} \end{cases}$$

↑ describes an alternate interpretation

$$|t_Z| \leq |t_Z^{\text{alt}}| = |1_C| - |t_Z| \Rightarrow |t_Z| \leq \frac{|C|}{2} \quad \square \text{ consistent w/ same charges.}$$

Thm: one syndrome enough to remove bad clusters...

$$\mathbb{P}[\text{error cluster of size } \geq S \text{ after decoding}] \leq n e^{-c's}$$

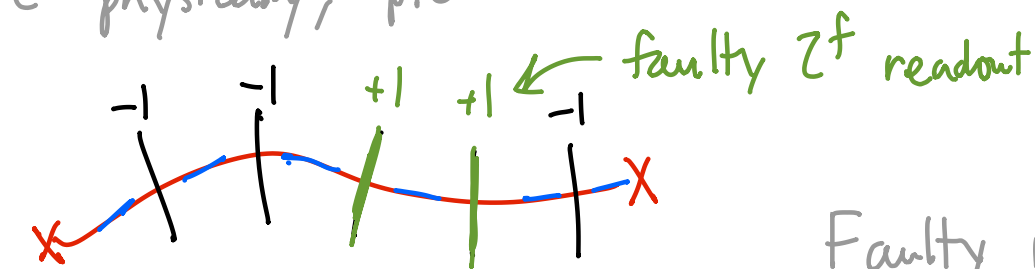
Proof sketch: connected component of t_2^{bad} of size S
 $\Rightarrow O(S)$ repair [by earlier Prop.]
 $S \ll d \Rightarrow$ after repair, residual errors are detectable errors

up to gauge syndromes!

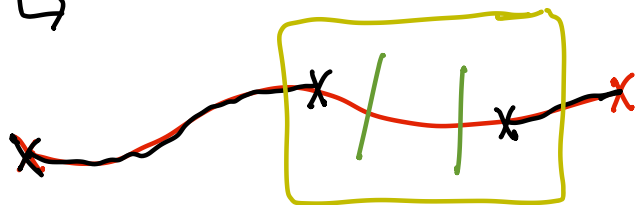
Analogous to earlier...

need $1/2$ -dense cluster of faulty Z^f to not detect long string.
 $\Rightarrow$ exponentially suppressed probability.

More physically, picture is:



[repaired to
→



Smaller error remains!

Faulty readout $\Rightarrow$
missing patches in valid
flux/charge lines.

Peierls $\Rightarrow$ safely repairable,
unlike 2d TC where we
could only measure syndromes.

Example of single-shot QEC which we'll explore more later...