

Recall chain complex formalism for understanding CSS codes...

$$Z\text{-checks} \xrightarrow{H_Z^T} \text{qubits} \xrightarrow{H_X} X\text{-checks}$$

$$\text{CSS commutation condition: } H_X H_Z^T = 0$$

Using connection to topology we also had the following:

$$\text{faces} \xrightarrow{\partial_2} \text{edges} \xrightarrow{\partial_1} \text{vertices}$$

toric
code

But suppose we had also used this chain complex to think about 2d Ising:

$$\text{Def: metachecks} \xrightarrow{M^T} \text{checks} \xrightarrow{H^T} \text{bits}$$

↳ consistency constraints b/w checks



$$(x_1 + x_2) + (x_2 + x_3) + (x_3 + x_4) + (x_4 + x_1) = 0$$

checks on edges

Possible Advantages of metacheck:

- ① thermodynamic stability (like 2d Ising)
- ② help w/ faulty syndrome (like 3d color)

Could we build a quantum code w/ X & Z metachecks?

$$Z\text{-meta} \xrightarrow{M_Z^T} Z\text{-check} \xrightarrow{H_Z^T} \text{qubits} \xrightarrow{H_X} X\text{-check} \xrightarrow{M_X} X\text{-meta}$$

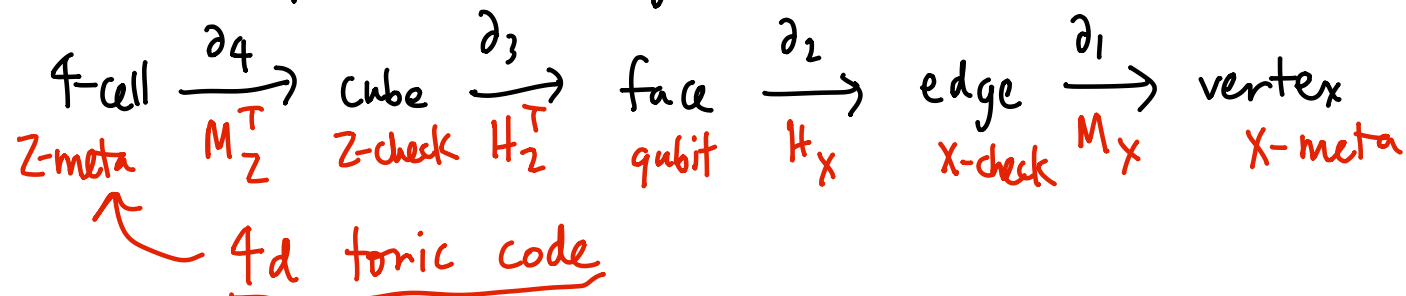
⇒ longer chain complex!

$$M_X H_X = 0 \quad M_Z H_Z = 0 \quad H_X H_Z^T = 0$$

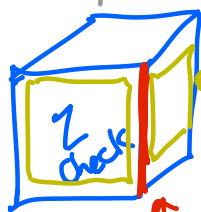
metacheck constraints

We can build these longer chain complexes by thinking about higher dimensional spaces! For our model above we'd need...

4-dim space. e.g. 4d torus:



To explicitly understand the nature of the 4d TC checks...



qubits on faces

(These 2 qubits shared btwn X & Z check...)

X-check on edge

For a general chain complex we have higher homology groups:

Def: n^{th} homology group: $H_n := \frac{\ker(\partial_n)}{\text{im}(\partial_{n+1})}$

$\Rightarrow$ non-trivial Z-logical classes for 4d toric code: $H_2 = \frac{\ker(\partial_2)}{\text{im}(\partial_3)}$

i.e. closed surfaces modulo the bdy of a 3d cube...

Prop: $\dim(H_2) = 6$ for 4d torus.

Proof sketch: Consider 4d space w/ periodic BCs...



$H_1 = \text{loop w/o bdy:}$

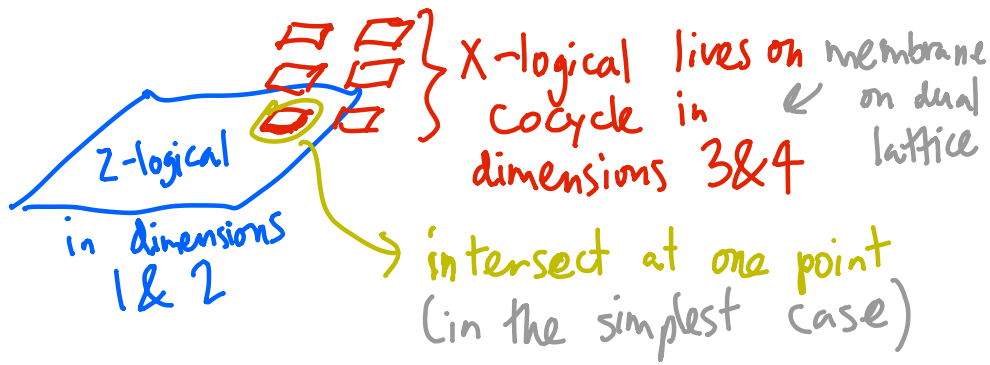
4 choices of dimension to wind around

$H_2 = \text{surface stretching 2 dims}$

$\binom{4}{2} = 6$ choices of 2 dimensions

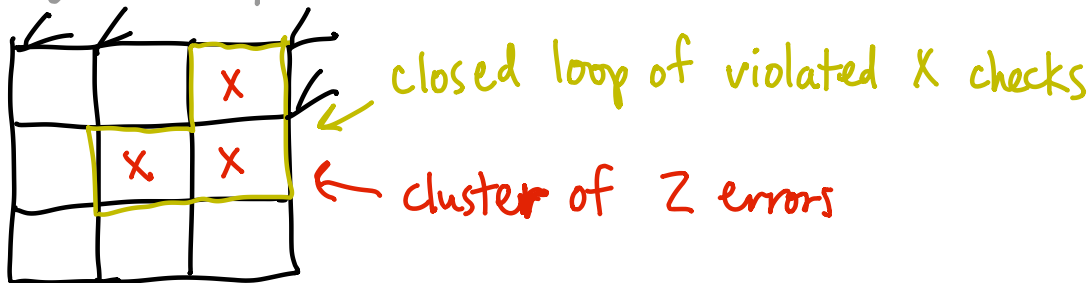
$\Rightarrow$ Z-logicals of 4d TC live on 6 non-trivial cycles

$\Rightarrow$ X-logicals on intersecting cocycles?



And just as before we can also deform the cycle/cocycle w/o affecting logical

Now let's think about excitations in 4d TC. Just like in 2d Ising, key point is that violated checks now form closed loops...

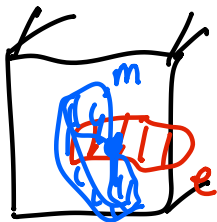


syndrome of cluster of X errors form closed loop on dual lattice but hard to draw this closed loop piercing through cubes in 4d.

Key conceptual picture to keep in mind is that

braiding of e & m anyons in 2d TC

→ braiding of e & m loops in 4d:



2d surfaces can intersect at a point in 4d
 $\Rightarrow$ braiding of loop excitations!

Now let's talk about the relevance of 4d TC for QEC.

Def: passive quantum memory / self-correction: a local "Gibbs sampler" decoder protects quantum info

$$\lim_{n \rightarrow \infty} t_{\text{mem}}(n) = \infty$$

Recall that 2d Ising had this property as a classical code.

To see why the 4d TC also has this property...

Recall: Gibbs state for quantum system: $\rho_\beta = \frac{1}{Z(\beta)} e^{-\beta H_0}$

$$H_0 = - \sum_{\text{edges } e} X^e - \sum_{\text{cubes } c} Z^c$$

$$\left. \begin{array}{l} \sum_{\text{edges } e} X^e \\ \sum_{\text{cubes } c} Z^c \end{array} \right\} \rightarrow \prod_{f \in \partial e} X_f \quad \prod_{f \in \partial c} Z_f$$

and all terms commute since CSS code.

X & Z syndromes behave analogously. Focus on detecting Z errors. same constant for any allowed η^e

Probability we see given X-syndrome $X^e = \eta^e$:

$$\text{tr} \left(\rho_\beta \prod_e \frac{I + \eta^e X^e}{2} \right) = \text{tr}_{\substack{\text{X-syndrome} \\ \text{logical}}} \left(e^{\beta \sum_e \eta^e} \text{tr}_{\substack{\text{Z-syndrome} \\ \text{logical}}} \left(\frac{e^{\beta \sum_c Z^c}}{Z(\beta)} \right) \right)$$

So up to this constant we may as well write:

$$\propto \sum_{\substack{x_f \in \{\pm 1\} \\ \text{classical spins}}} \frac{1}{Z_x(\beta)} e^{\sum_e \beta \eta^e} \delta_{\eta^e, \prod_{f \in \partial e} x_f}$$

classical partition function

Key point is we can think about a classical stat mech problem.

This classical problem is similar to the MWPM Hamiltonian we saw earlier but here we're interested in the thermal ensemble, not just the ground state.

A Z error acts to flip $x_f \rightarrow -x_f$.

What would a Gibbs sampler do here? Using our Metropolis update rule from before

$$P[\text{flip } x_f \rightarrow -x_f] = \min(1, e^{-\beta[E(-x_f, \dots) - E(x_f, \dots)]})$$

where $E(\{x_f\}) = - \sum_{\text{edge } e} \prod_{f \in \partial e} x_f$

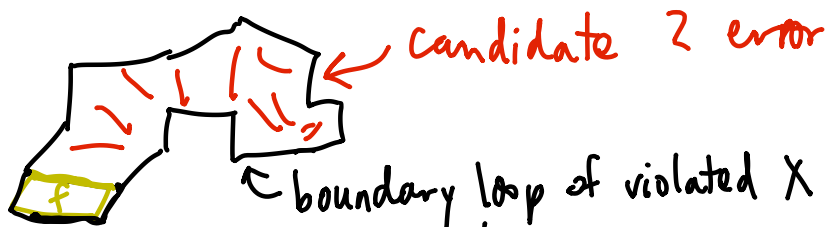
$$E(-x_f, \dots) - E(x_f, \dots) = 2 \sum_{e \in \partial f} x_f \prod_{\substack{f' \in e \\ f' \neq f}} x_{f'}$$

This is computable locally; update rule doesn't require global data.
Just like in 2d Ising model.

Thm: this decoder protects logical information for time

$$t_{\text{mem}} \gtrsim e^{\alpha L} \quad \text{if } \beta > \beta_c.$$

Proof sketch: follows what we did for 2d Ising model



closed b/c of vertex metachecks!

Peierls estimate: how many closed loops contain face f?

each vertex has $2 \cdot 4 = 8$ adjacent edges $\Rightarrow$ $< 7^L$ loops of length L pass through given edge (lazy count)

Energy cost: $\Delta E = 2L \Rightarrow \mathbb{P}[\text{loop of length } L \text{ through one vertex}] \sim (7e^{-2\beta})^L$

decays exponentially w/ L .

Largest such loop anywhere: $n(7e^{-\beta})^{L_*} \sim 1 \Rightarrow \underline{L_* \sim \log n}$

Now let's talk about 4d TC parameters on $L \times L \times L \times L$ torus:

$$\hookrightarrow n = \# \text{ of qubits} = \# \text{ of faces} = 6L^4 \quad k=6$$

code distance $d = L^2 = \min(\# \text{ of faces on noncontractible cycle})$

Now just like for 2d Ising model,

to nucleate logical error, need to grow large logical error...



also valid
error for
same syndrome

partial error has perimeter $\gtrsim L$ of violated checks
 $\Rightarrow$ free energy barrier $\Delta F \sim L$

Bottleneck Thm $\Rightarrow t_{\text{mem}} \gtrsim e^{\beta \Delta F} \sim e^{\beta L}$ assuming starting from typical configuration!

Note that we do not need to identify exact error... only error up to Z-check equivalent choice.

Key point is that no small cluster of violated checks has ambiguous decoding $\Rightarrow$ all separately correctable errors.

Logical error requires crossing free energy barrier w/ many violated checks!

Notice: thermal decoder does NOT return us to codespace
Just needs to leave us w/ another easily correctable error...