

Let's return to 4d toric code decoding and think about why 4d toric code is nice from perspective of fault-tolerance...

4-cells $\xrightarrow[\text{M}_Z^T]{\partial_4}$ cubes $\xrightarrow[\text{H}_Z^T]{\partial_3}$ faces $\xrightarrow[\text{H}_X]{\partial_2}$ edges $\xrightarrow[\text{M}_X]{\partial_1}$ vertices

We now recall the problem of fault-tolerant quantum memory...

Z-errors $\longrightarrow$ X-check syndrome extraction

e_Z Talk about Z-errors since u_X X-errors handled analogously
observed X-syndrome: $u_X = \underbrace{H_X e_Z}_{\text{actual syndrome}} + t_X$ faulty measurement.

Now use metachecks (or fact loops must be closed!)

$$M_X u_X = M_X H_X e_Z + M_X t_X \Rightarrow \text{metachecks hint where faults occur}$$

Now the idea is very similar to 3d gauge color code so we elide the details...

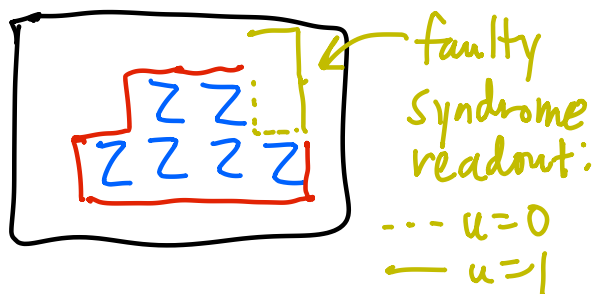
Decode: (1) choose minimum weight error $\hat{u}_X$ w/ $M_X \hat{u}_X = M_X u_X$

(2) assume valid syndrome was $s_X = u_X + \hat{u}_X$ ($M_X s_X = 0$)

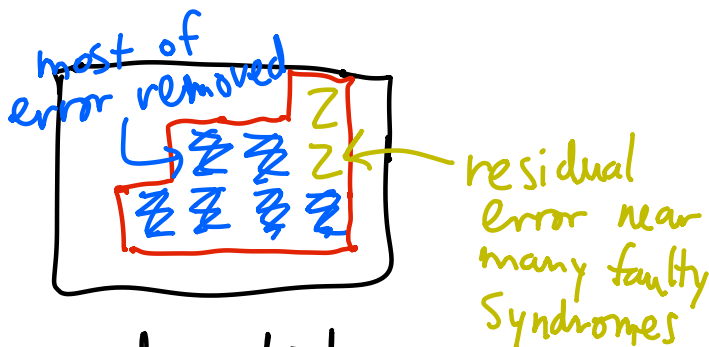
correct small error e_Z w/ $s_X = H_X e_Z$. along its boundary
caveat: homology of torus relevant for large enough clusters...

Prop: Residual error cluster had $\geq 50\%$ of faulty syndromes

Proof omitted as we did analogous thing for 3d gauge color code



observed



decoded

residual error near many faulty syndromes

Just like we've seen before, we further have in an independent error model that

$$P[\text{faulty syndrome cluster of size } L] \lesssim [\alpha \cdot p_m]^{L/2} \cdot n$$

if qLDPC

$L/2 \leftarrow \# \text{ of syndromes need error}$
 $n \leftarrow \# \text{ of qubits to root error at}$

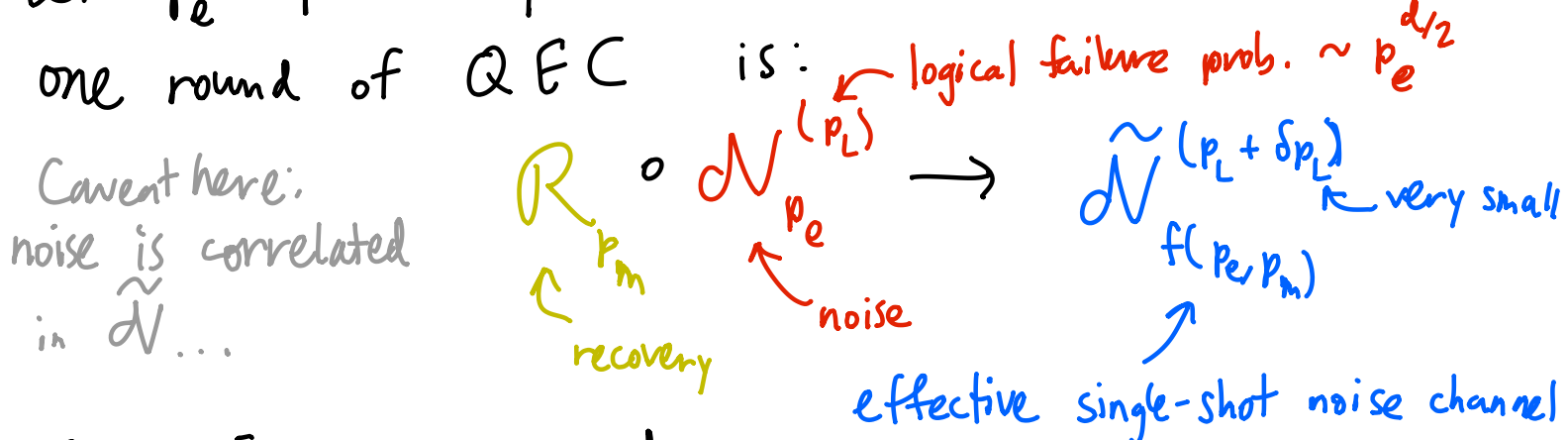
$\Rightarrow$ negligible probability for uncorrectable error [$L \gtrsim L \sim \sqrt{d}$]
 if $d \gg \log n$.

Indeed, 4d TC has $[[6L^4, 6, L^2]]$ so $d \sim \sqrt{n}$ works

Key difference w/ 2d TC:
 error suppression w/ single syndrome
 So we do NOT need spacetime decoding!

Def: single-shot QEC: one round of noisy syndromes suffices to control error growth.

let p_e = prob of qubit error p_m = faulty syndrome probability



Caveat here:
 noise is correlated in $\tilde{\mathcal{N}} \dots$

$$Sp_L \lesssim [O(1) \cdot \max(p_e, p_m)]^{d/2}$$

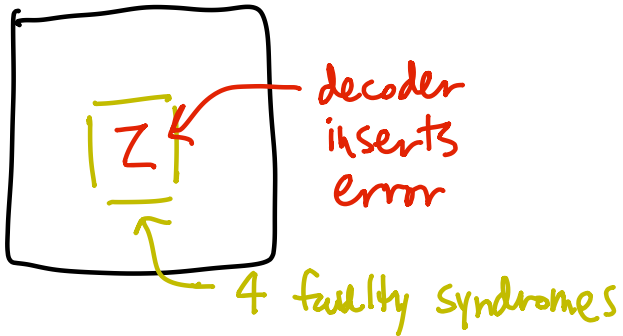
[so many faulty syndromes the decoder causes uncorrectable error]

Iterating the single-shot recovery map many times...

$$\underbrace{R_{p_m} \circ \mathcal{N}_{p_e}^{(p_L)} \circ \dots \circ R_{p_m} \circ \mathcal{N}_{p_e}^{(p_L)}}_{T \gg 1 \text{ rounds}} \rightarrow \mathcal{N}_{\tilde{f}(p_e, p_m)}^{(Tp_L + TSp_L)} \quad (\text{schematic})$$

$\Rightarrow$ quantum memory for $T \ll \frac{1}{p_L + \delta p_L} \sim \exp(L)$ as $L \rightarrow \infty$

Note: $\tilde{F}(p_e, p_m)$ does NOT approach 0 for fixed p_e, p_m .



for any LDPC code we return to finite error density state.

OK if these errors are correctable.

Experiment reads out final state by repeated single-site X_i meas.
 $\rightarrow$ collapse state to logical $\bar{X} = \pm 1$ (e.g.)

We just need this final set of measurements to detect/correct (in software) only correctable Z errors, which is what single-shot QEC does.

So to summarize, codes and decoders studied thus far have:

code	single-shot	local decoder
2d SC	N	N $\leftarrow$ one is constructed in literature but is very complicated
3d gauge color	Y	N
4d TC	Y	Y

this combination enables MFQEC
 "measurement-free" QEC

Name is potentially misleading... MFQEC is very similar in spirit to regular QEC but the way it's implemented experimentally is different and can have some advantages...

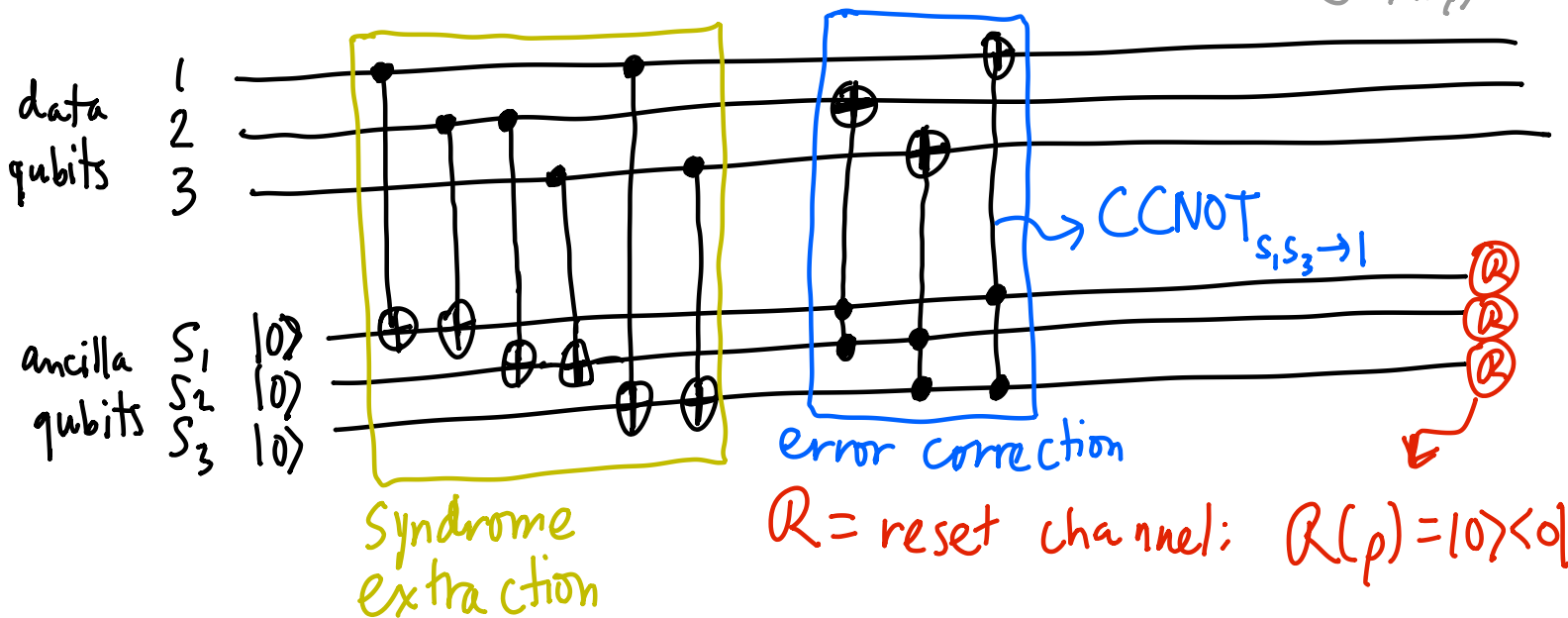
We will motivate MFQEC by looking at a simple example.

Take 3-qubit repetition code; stabilizers

$$S_1 = Z_1 Z_2, \quad S_2 = Z_2 Z_3, \quad S_3 = Z_1 Z_3 \quad [\text{redundant set}].$$

MFQEC

circuit: (not optimized! drawn for clarity)



Def: CCNOT gate;

$\text{CCNOT}_{12 \rightarrow 3} |x_1, x_2, x_3\rangle = \begin{cases} |x_1, x_2, (1-x_3)\rangle & x_1 = x_2 = 1 \\ |x_1, x_2, x_3\rangle & \text{else} \end{cases}$

Z-eigenbasis
↓

More generally...

$$C^{(l)} \text{NOT}_{1 \dots l \rightarrow l+1} |x_1 \dots x_{l+1}\rangle = \begin{cases} |x_1 \dots (1-x_{l+1})\rangle & x_1 = \dots = x_l = 1 \\ |x_1 \dots x_{l+1}\rangle & \text{else} \end{cases}$$

We already saw how the first part of the circuit extracts the Syndrome. Why does second part work?

Take initial state: $[\alpha + \beta X_1] \underbrace{|000\rangle}_{\text{codeword on data}} \otimes \underbrace{|000\rangle}_{\text{ancilla record flagged checks}} = |\psi_0\rangle$

Correctable error

After syndrome extraction: $|\psi_1\rangle = \alpha |000\rangle \otimes |000\rangle + \beta |100\rangle \otimes |101\rangle$

Now: $CCNOT_{s_1 s_2 \rightarrow 2} |100\rangle \otimes \underbrace{|101\rangle}_{\substack{\downarrow \\ \text{not both } 1}} = |100\rangle \otimes |101\rangle$

$CCNOT_{s_1 s_3 \rightarrow 1} |100\rangle \otimes \underbrace{|101\rangle}_{\substack{\downarrow \\ \text{both } 1}} = X_1 |100\rangle \otimes |101\rangle = |000\rangle \otimes |101\rangle$

After $CCNOT_5$: $|\psi_2\rangle = |000\rangle \otimes [\alpha|000\rangle + \beta|101\rangle]$

error info entirely in ancilla Hilbert space.

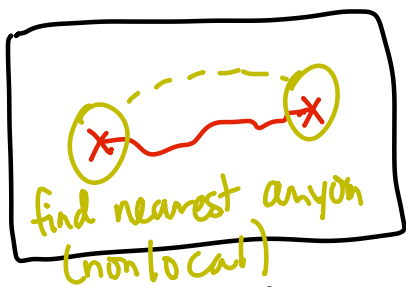
Reset channel: $R(|\psi_2\rangle\langle\psi_2|) = \underbrace{|000\rangle\langle 000|}_{\substack{\swarrow \\ \text{returned to codespace}}} \otimes \underbrace{|000\rangle\langle 000|}_{\substack{\swarrow \\ \text{fresh ancilla.}}}$

This is essentially a physical realization of Stinespring's formulation of quantum measurement: measurement is unitary entangling of system w/ a qubit that stores the measurement outcome! QEC is then applying feedback conditioned on that outcome back onto original qubits!

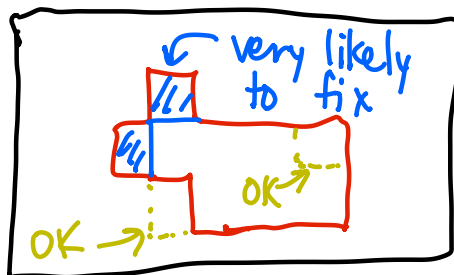
Advantage of MFQEC is experimental:

reset may be easier (e.g. $\sim 100\times$ faster in neutral atoms)

Challenge is that syndrome extraction and feedback circuits are usually not local.



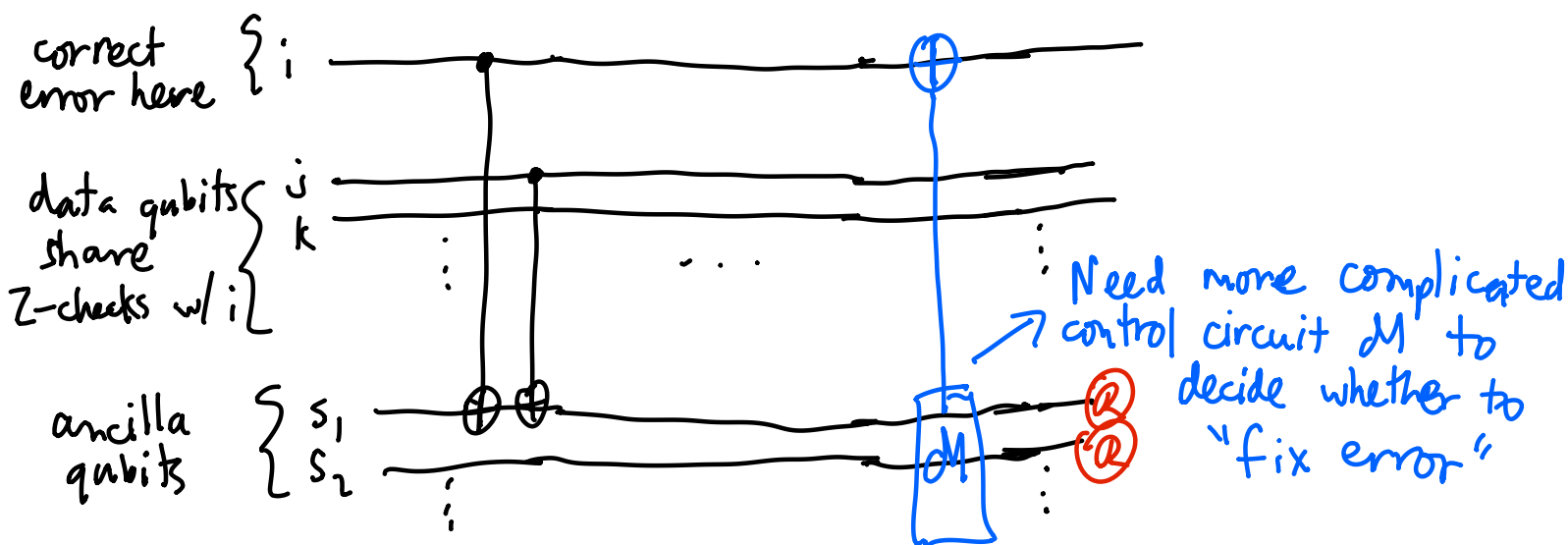
2d surface code



4d toric code

But 4d TC has guarantee that local decoder can fix errors w/ high probability - likely to stop syndrome membrane from growing too much (wherever lots of curvature, remove it).

Formal (if "inefficient") strategy for MFQEC Gibbs decoder...



Minimal choice: greedy decoder: M does majority vote (does this reduce # of violated checks)

Gibbs sampler: apply "partial" errors?

$$\begin{aligned}
 M|x_i, s_1 \dots s_p\rangle \otimes |0\rangle &= \delta_{s_{r,1}} \dots \delta_{s_{p,1}} |1-x_i, s_1 \dots s_p\rangle \otimes |0\rangle \\
 &+ \delta_{s_{1,0}} \dots \delta_{s_{p,0}} \sqrt{1-e^{-\Delta E}} |x_i, s_1 \dots s_p\rangle \otimes |0\rangle \\
 &+ \delta_{s_{1,0}} \dots \delta_{s_{p,0}} \sqrt{e^{-\Delta E}} |1-x_i, s_1 \dots s_p\rangle \otimes |1\rangle + \dots
 \end{aligned}$$

← ancilla for Gibbs sampler

Such decoders usually have poor thresholds, but they are much closer to mimicking how physics (heat bath) would protect an ordered phase of matter!