Can we find qLDPC code w/ $[[n, \Theta(n), \Theta(n)]]$

asymptotically good

Random codes did this classically. Quantumly we'll need a few more ingredients...

Question: alternatives to homology for constructing $H_X H_Z^T = 0$?

Simple idea: $H_X = (A \quad B)$ $\quad H_Z = (B^T \quad A^T)$

take 2 $m \times m$ matrices in $\mathbb{F}_2^n$

$$\Rightarrow H_X H_Z^T = (A \quad B)\begin{pmatrix} B \\ A \end{pmatrix} = AB + BA = 0?$$

if A and B commute

Choose A, B so code is non-trivial? $\ker(H_X) \neq \text{im}(H_Z^T)$.

Idea #1? Take $B = I$ so $2A = 0$ automatically in $\mathbb{F}_2$?

Problem: $\text{rank}(H_X) = \text{rank}((A \quad I)) = m = \text{rank}(H_Z)$.

$\Rightarrow k = n - \text{rank}(H_X) - \text{rank}(H_Z) = 2m - m - m = 0$.

more generally: $k = [m - \text{rank}(H_X)] + [m - \text{rank}(H_Z)]$

$\Rightarrow$ check redundancy essential to get codewords!

Prop: $k = \dim(\ker A \cap \ker B) + \dim(\ker(A^T) \cap \ker(B^T))$

Proof: Calculate rank by multiplying on the left

$$s^T H_X = 0 \Rightarrow A^T s = B^T s = 0 \Rightarrow s \in \ker(A^T) \cap \ker(B^T)$$

$$\Rightarrow m - \text{rank}(H_X) = \dim(\ker(A^T) \cap \ker(B^T))$$

The other term comes from an analogous calculation for H_Z ∎

Idea #2? Choose $A=B$ so $2A^2=0$?

But then: if $\text{rank}(A) < n$, then $\exists$ u w/ $|u|=1$ and $u \notin \text{im}(A^T)$.

$\Rightarrow \begin{pmatrix} u \\ u \end{pmatrix} \in \ker(H_2)$ is weight-2 non-trivial logical $\Rightarrow d \leq 2$

One mechanism that does generally work is:

$A = I \otimes H_2$ and $B = H_1^T \otimes I$ for any H_1 & H_2

$\Rightarrow$ HGP code. So we need to look for something better...

We need $AB=BA$ to hold automatically, but also for A&B to be non-trivial. Ideas from group theory can prove very useful...

Def: <u>regular representations</u> of group G:

Consider $|G|$-dimensional vector space

$V := \text{span}\{|g\rangle\}_{g \in G}$ [each group element labels a state]

left representation:

$L_h |g\rangle := |hg\rangle$

right representation:

$R_h |g\rangle = |gh^{-1}\rangle$

Check that: $L_{h_1} L_{h_2} = L_{h_1 h_2}$ $\quad$ $R_{h_1} R_{h_2} = R_{h_1 h_2}$

Crucially: $L_{h_1} R_{h_2} |g\rangle = |h_1 g h_2^{-1}\rangle = R_{h_2} L_{h_1} |g\rangle$

$\Rightarrow$ L & R commute even though LL/RR do NOT commute for a non-Abelian group.

Def: often refer to V as <u>group algebra</u>:

$$V \to \mathbb{F}_2[G] := \left\{ \sum_{g \in G} \underbrace{c_g}_{c_g \in \mathbb{F}_2} |g\rangle \to \sum_{g \in G} c_g \, g \right\}$$

We often omit vector symbols and just think of formal sums of group elements. This is useful because we can then write:

$$\left(\sum_g c_g\, g\right)\left(\sum_{g'} d_{g'} g'\right) = \sum_g \left(\sum_h c_h d_{h^{-1}g}\right) g$$

$\Rightarrow$ group algebra comes w/ left/right multiplication.

Def: 2-block group algebra (2BGA) code:

$$H_X = (L_a \quad R_b) \qquad H_Z = (R_b^T \quad L_a^T)$$

where $a, b \in \mathbb{F}_2[G]$ are elements of group algebra.

Here $L_a = \sum_{g\in G} c_g L_g \iff a = \sum_{g\in G} c_g\, g$

2BGA is LDPC $\iff$ a & b are sums of $O(1)$ group elements.

2BGA codes can be studied using both Abelian/non-Abelian G. First it makes sense to focus on Abelian G as there are already some modern codes of great interest we can discover!

Def: cyclic group $\mathbb{Z}_m = \{1, g, g^2, \dots, g^{m-1}\}$
$= $ group of integers (mod m) under addition.
$\Rightarrow \mathbb{Z}_m$ is Abelian

Def: product group: $G_1 \times G_2$ is a group where
$(g_1, g_2)\cdot(h_1, h_2) = (g_1 h_1, g_2 h_2)$

Thm: G = finite Abelian group $\iff G = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \mathbb{Z}_{m_p}$.

So it suffices to think about codes over cyclic groups.

Focus on case $G = \mathbb{Z}_\ell \times \mathbb{Z}_m$.

$= \{(x^i, y^j)\}_{0 \le i < \ell,\ 0 \le j < m}$

Note: Abelian $G \Rightarrow L_g = R_{g^{-1}}$ (write $L_g \to g$)

Example: 2d toric code comes from choosing

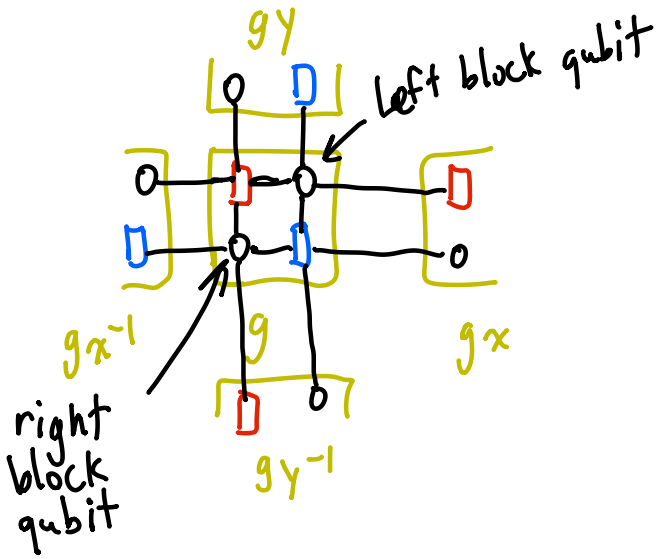
$$A = 1 + x \qquad B = 1 + y^{-1}$$

$$\langle h' | L_g^T | h \rangle = \langle gh' | h \rangle = \delta_{gh',h} = \delta_{h',g^{-1}h} = \langle h' | L_{g^{-1}} | h \rangle$$

$$\Rightarrow H_X = \begin{pmatrix} (I + x) & (I + y^{-1}) \end{pmatrix} \qquad H_Z = \begin{pmatrix} (I + y) & (I + x^{-1}) \end{pmatrix}$$

I & x are $|G| \times |G|$ matrices w/ $|G| = \ell m$

To illustrate this is 2d TC useful to draw out:



which is precisely the layout of the 2d TC (up to asymmetric placement of bits/checks in Tanner graph!)

Useful interpretation: x & y are discrete translations along directions

Now let's look for a code beyond 2d TC using same group...

Def: Bivariate bicycle (BB) code: 2BGA w/ $G = \mathbb{Z}_\ell \times \mathbb{Z}_m$ and weight-3 A & B.

These codes were developed by IBM and are actively exptly pursued!

Example: $\ell=12,\ m=6$: $A = x^3 + y + y^2$ $\quad B = y^3 + x + x^2$

$\Rightarrow$ LDPC w/ weight-6 checks. Numerical calculation shows that code parameters are $[[144, 12, 12]]$

compare to rotated surface code: ↕ huge physical (data) qubit savings

$[[L^2, 1, L]] \Rightarrow [[144 \times 12, 12, 12]]$ using decoupled SC.

(pseudo) threshold under circuit-level noise: $p \approx 0.007$ vs. 0.01 for SC — both well within expt'l reach.

BB codes (like most high-rate: large k/n) codes have few transversal gates.

↳ ① some gates from qubit permutations:

L/R g-qubit $\longrightarrow$ L/R (hg)-qubit

preserves stabilizer group: $H_X \to h H_X h^{-1} = H_X$.

② global "Hadamard": $\bar{H} = H^{\otimes 144} \cdot \Sigma$

where Σ swaps qubits: $\Sigma X_{(i,j)_{L/R}} \Sigma^{-1} = X_{(-i,-j)_{R/L}}$ etc. (same for Y & Z)

Sends group algebra element $g \to g^{-1}$ or $A \to A^T$ in matrix form

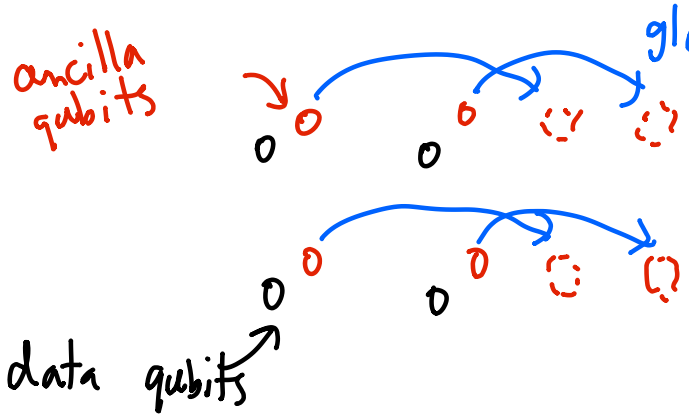
because $\bar{H}$ sends $H_X \to H_Z \cdot M_\Sigma^T = (B^T \ A^T) \begin{pmatrix} 0 & \text{inv} \\ \text{inv} & 0 \end{pmatrix}$

$= (A \ B) = H_X$

where $\text{inv} \cdot A = JAJ$, where $J|g\rangle = |g^{-1}\rangle$

$\bar{H}$ swaps logical qubits along w/ exchanging X & Z type logicals

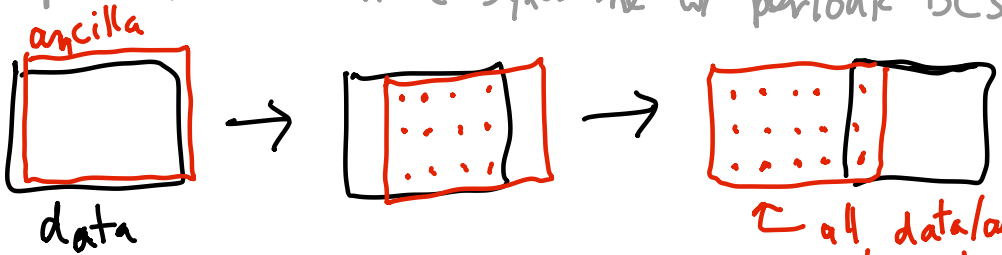
BB codes also give a useful illustration of code that's well-matched to neutral atom hardware.



global translation of atom arrays
⇒
translate ancilla by $x^a y^b$

Using movable optical tweezer arrays, translating ancilla qubits over data qubits gives efficient syndrome extraction.

To implement the entire syndrome w/ periodic BCs:



↳ all data/ancilla interactions achieved via local interactions

In contrast w/ superconducting circuit arrays one needs to add a lot of extra wires and couplers!

What if we increase ℓ & m to be large?

↳ BB codes (w/ growing distance) "look like" copies of 2d TC

⇒ constrained by BPT [$k \sim O(1)$]

Details of k & d can be sensitive to ℓ & m w/ fixed A & B.

↳ if A & B have generators that grow w/ n ⇒ not geometrically local in 2D but is local in

$$D \leq \Delta_c - 2$$

where Δ_c is check weight