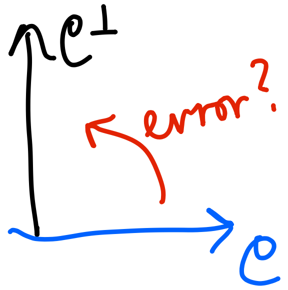
We've seen a specific example of a quantum code. Today we present a general/abstract theory of quantum codes. Actually the abstraction is worth it — we'll see that (relatively) simple conditions guarantee protection against <u>all</u> quantum errors.

Let $\mathcal{H}$ be a (discrete) Hilbert space.

$\mathcal{C} \subseteq \mathcal{H}$ is the codespace we want to protect...



P = projector onto $\mathcal{C}$.

<u>Goal of QEC:</u> if $\rho = P\rho P$, then

decoder $(\mathcal{R} \circ \mathcal{N})(\rho) = \rho$

for arbitrary $\mathcal{N}(\rho) = \sum_i K_i \rho K_i^\dagger$ if $\underbrace{K_i \in \text{span}\{\text{correctable error}\}}_{\text{e.g. not just } X_1 \,\&\, Z_1\text{, but } \alpha X_1 + \beta Z_1 \cdots}$

<u>Thm:</u> QEC if and only if, for basis $\{E_a\}$ of correctable errors,

$$P E_a^\dagger E_b P = \alpha_{ab} P \quad \text{(Knill-Laflamme condition)}$$

α Hermitian: $\alpha_{ab} = \overline{\alpha_{ba}}$

Proving this theorem will be the rest of the lecture. We'll stop and comment on the implications as we go!

WLOG, choose basis so that:

$$\alpha_{ab} \to \begin{pmatrix} \delta_{\tilde{a}\tilde{b}} & 0 \\ 0 & 0 \end{pmatrix}$$

(diagonalize α, then rescale...)

Some E's may annihilate codespace! We'll return to that...

Lemma 1: Build $\mathcal{R}$ for simplified $\mathcal{N}(\rho) = E_{\tilde{a}} \rho E_{\tilde{a}}^\dagger$ (for any $\tilde{a}$).

Proof: if $|\psi\rangle, |\phi\rangle \in \mathcal{C}$, then:

$$\langle \psi | \phi \rangle = \langle \psi | P | \phi \rangle = \langle \psi | P E_{\tilde{a}}^\dagger \underbrace{E_{\tilde{a}} P | \phi \rangle}$$

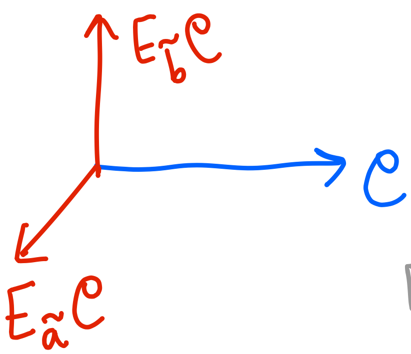
So $E_{\tilde{a}} P$ "acts like unitary" on $\mathcal{C}$:

there exists ↳ $\exists$ unitary $U_{\tilde{a}}$ w/ $E_{\tilde{a}} P = U_{\tilde{a}} P$.

Define rotated projectors: $U_{\tilde{a}} P U_{\tilde{a}}^\dagger = P_{\tilde{a}}$.

$$KL \Rightarrow \quad P_{\tilde{b}} P_{\tilde{a}} = U_{\tilde{b}} P \underbrace{\cancel{U_{\tilde{b}}^\dagger U_{\tilde{a}}}}_{E_{\tilde{b}}^\dagger E_{\tilde{a}}} P U_{\tilde{a}}^\dagger = \delta_{\tilde{a}\tilde{b}} P_{\tilde{a}}.$$

WLOG assume one $E_{\tilde{a}} = I$ (identity)... then,



Each detectable $\tilde{E}_a$ maps to orthogonal states!

Exactly like what happened in Shor code!

Claim: decoder $\mathcal{R}(\rho) = \sum_{\tilde{a}} U_{\tilde{a}}^{-1} P_{\tilde{a}} \rho P_{\tilde{a}} U_{\tilde{a}} + P_\perp \rho P_\perp$

where $P_\perp = I - \sum_{\tilde{a}} P_{\tilde{a}}$ projects onto "inaccessible" states.

→ measure which syndrome sector $\tilde{a}$

→ rotate back to code space!

$\mathcal{R}$ is a quantum channel since:

$$\sum_{\tilde{a}} P_{\tilde{a}} U_{\tilde{a}} U_{\tilde{a}}^{-1} P_{\tilde{a}} + P_\perp = I$$

$$\mathcal{R}(E_{\tilde a} \underbrace{P \rho P}_{\text{must start in codespace!}} E_{\tilde a}^\dagger) = \sum_{\tilde b} \underbrace{U_{\tilde b}^{-1} P_{\tilde b} E_{\tilde a} P}_{= P U_{\tilde b}^{-1} E_{\tilde a} P = P E_{\tilde b}^\dagger E_{\tilde a} P = \delta_{\tilde a \tilde b} P} \rho P E_{\tilde a}^\dagger P_{\tilde b} U_{\tilde b}$$

$$= \sum_{\tilde b} \delta_{\tilde a \tilde b} P \rho P = P \rho P \qquad \blacksquare$$

Lemma 2: $\mathcal{R}$ actually corrects arbitrary $\mathcal{N}(\rho) = \sum K_i \rho K_i^\dagger$ if $K_i \in \operatorname{span}\{E_{\tilde a}\}$.

Proof: Linearity $\Rightarrow$ check for $\mathcal{N}(\rho) = K \rho K^\dagger$ w/ $K = \sum k_{\tilde a} E_{\tilde a}$.

It's basically now the same calculation because:

$$\sum_{\tilde b} P E_{\tilde b}^\dagger \Big(\sum_{\tilde a} k_{\tilde a} E_{\tilde a}\Big) P \rho P \Big(\sum_{\tilde a'} \overline{k_{\tilde a'}} E_{\tilde a'}^\dagger\Big) E_b P$$

$$= \sum_{\tilde a, \tilde a', \tilde b} k_{\tilde a} \overline{k_{\tilde a'}} \delta_{\tilde a \tilde b} \delta_{\tilde a' \tilde b} P \rho P \quad \propto \quad P \rho P$$

Since $\mathcal{R}$ is a channel, if $\mathcal{N}$ is channel, $\mathcal{R} \circ \mathcal{N} = \text{CPTP}$

$\Rightarrow (\mathcal{R} \circ \mathcal{N})(P\rho P) = P \rho P.$ $\blacksquare$

This went fast – the physics here is critical. Notice:

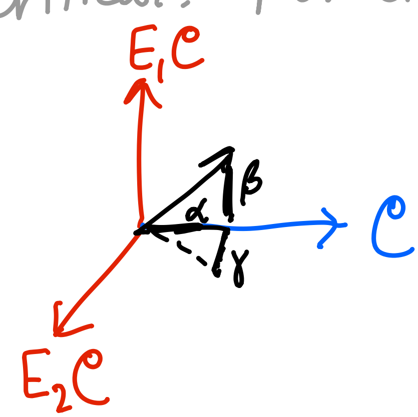
error $(\alpha I + \beta E_1 + \gamma E_2)|\psi\rangle_{\in \mathcal{C}}$:

when measuring syndrome:

$\rightarrow P_{\mathcal{C}}$ w/ prob $|\alpha|^2$

P_1 ... $|\beta|^2$

P_2 ... $|\gamma|^2$



Measurement collapsed state to : $\begin{cases} |\psi\rangle \\ E_1|\psi\rangle \\ E_2|\psi\rangle \end{cases}$

Now lemma 1 $\Rightarrow$ $\mathcal{R}$ works.

In this respect, quantum error correction is similar to digital classical error correction $\Rightarrow$ only fix discrete set of errors. Continuum of possible errors affects probability of detecting a syndrome only!

Lemma 3: $\mathcal{R}$ works if we include errors $E_{\bar{a}}$ w/

$$P E_{\bar{a}}^{\dagger} E_{\bar{b}} P = 0.$$

Proof: Linearity $\Rightarrow$ take $K = \underbrace{\sum_{\tilde{a}} k_{\tilde{a}} E_{\tilde{a}}}_{\tilde{K}} + \sum_{\bar{a}} k_{\bar{a}} E_{\bar{a}}$

But: $KP = \tilde{K} P \Rightarrow$ reduces to previous case! $\blacksquare$

Common $E_{\bar{a}}$ error: $\tilde{E}_a (S - I)$

stabilizer!

e.g. Shor code could correct: $\underbrace{Z_1}_{\tilde{a}}, \underbrace{Z_2, Z_3}_{\text{"}\bar{a}\text{"}}$

(but they're not distinguishable)

since $Z_2 = Z_1(Z_1 Z_2 - I) + Z_1$

These remarks complete the first part of our Thm: KL conditions imply QEC. Now we need to show that QEC $\Rightarrow$ KL, or equivalently, not KL $\Rightarrow$ not QEC.

Lemma 4: if $(\tilde{R} \circ \mathcal{N})(P\rho P) = P\rho P$ and $\mathcal{N}(\rho) = \sum_i K_i \rho K_i^\dagger$

for some decoder:

then KL obeyed: $P K_j^\dagger K_i P = \alpha_{ji} P$.

Proof: if $\tilde{R}(\rho) = \sum_\mu D_\mu \rho D_\mu^\dagger$, then:

$$(\tilde{R} \circ \mathcal{N})(P\rho P) = \sum_{\mu, i} D_\mu K_i P \rho P K_i^\dagger D_\mu^\dagger = P\rho P.$$

Claim: this implies $D_\mu K_i P = c_{\mu i} P$ ← constants.

Reason: $\langle \phi | (\tilde{R} \circ \mathcal{N} - \mathcal{I})(|\psi\rangle\langle\psi|) | \phi \rangle = \sum_{\mu, i} |\langle \phi | D_\mu K_i | \psi \rangle|^2 - |\langle \phi | \psi \rangle|^2 = 0.$

if $|\psi\rangle \in \mathcal{C}$.

So if $\langle \phi | \psi \rangle = 0$ we must have $\langle \phi | D_\mu K_i | \psi \rangle = 0 \ \forall \mu, i$.

for all

$\Rightarrow D_\mu K_i \propto I$

Since $|\psi\rangle \in \mathcal{C} \Rightarrow D_\mu K_i P \propto P$.

Now because R was CPTP we know that:

$$\sum_\mu P K_i^\dagger D_\mu^\dagger D_\mu K_j P = \sum_\mu P \overline{c_{\mu i}} c_{\mu j} P = \alpha_{ij} P$$

where $\alpha_{ij} := \sum_\mu \overline{c_{\mu i}} c_{\mu j}$. ∎

This was precisely the KL condition which completes the proof of our theorem!

More intuition for lemma 4:

If there was any E where, for $|\psi\rangle \perp |\phi\rangle \in \mathcal{C}$,

$$\langle\psi|E|\psi\rangle \neq \langle\phi|E|\phi\rangle,$$

measuring E + measuring no syndrome (projector $P_{\mathcal{C}}$)

$$\text{collapses logical } \alpha|\psi\rangle + \beta|\phi\rangle \longrightarrow \begin{cases} |\psi\rangle & \text{prob. } |\alpha|^2 \\ |\phi\rangle & \text{prob. } |\beta|^2 \end{cases}$$

So we can only correct against errors which don't do this!