Goal: code performance improves as $n \to \infty$

# of physical qubits

→ protects against more errors
→ decoding easier?

Analogies to physics provide insight to answering these questions.

Today we'll study a minimal example of a scalable code. It has very clear connections to stat mech that foreshadow many big ideas...

Classical repetition code:

$$\text{logical} \quad \begin{matrix} 0 \leftarrow 00\cdots0 \\ 1 \leftarrow \underbrace{11\cdots1}_{n \text{ physical bits.}} \end{matrix}$$

Previously we took $n=3$. Can take larger n.

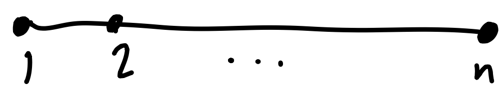
These look like the ground states of a phase of matter!

Map: $000 \to +++$ and $111 \to ---$

$\Rightarrow$ ferromagnetic ground states!

Let $z_i \in \{\pm 1\}$ denote classical variables. Consider 1d Ising model:

$$H = \sum_{i=1}^{n-1} (-z_i z_{i+1})$$



Ground states have $(z_i z_{i+1} = 1)$ for each i

multiply by z_{i+1} : $\boxed{z_i = z_{i+1}}$ $\Rightarrow$ repetition code

From a physics perspective, the 2 ground states are related by:

$\boxed{\mathbb{Z}_2 \text{ symmetry:}}$ $(z_1, \ldots, z_n) \longrightarrow (-z_1, \ldots, -z_n)$

takes $H \to H$, since $z_i z_{i+1} \to (-z_i)(-z_{i+1}) = z_i z_{i+1}$

→ Spontaneously broken: ground states not invariant.

So far this is translating repetition code into a physics language. Let's now return to error-correction perspective for a bit and ask: what's the point of using $n \gg 3$ bits?

Claim: $n \to \infty$ limit $\Rightarrow$ very robust against errors?

Independent error model:

$$z_i \longrightarrow \eta_i z_i$$

where $\mathbb{P}(\eta_i = +1) = 1-p$
$\mathbb{P}(\eta_i = -1) = p$
probability of bit flip

Let's ask how many errors we expect to see:

$$e_i = \frac{1-\eta_i}{2} = \begin{cases} 0 & \text{no error} \\ 1 & \text{error} \end{cases}$$

So # of errors $= N = \sum_{i=1}^{n} e_i$

$$\underbrace{\mathbb{E}}_{\text{expectation value}}[N] = \sum_{i=1}^{n} \mathbb{E}[e_i] = n \cdot \left[(1-p)\cdot 0 + p \cdot 1\right] = pn$$

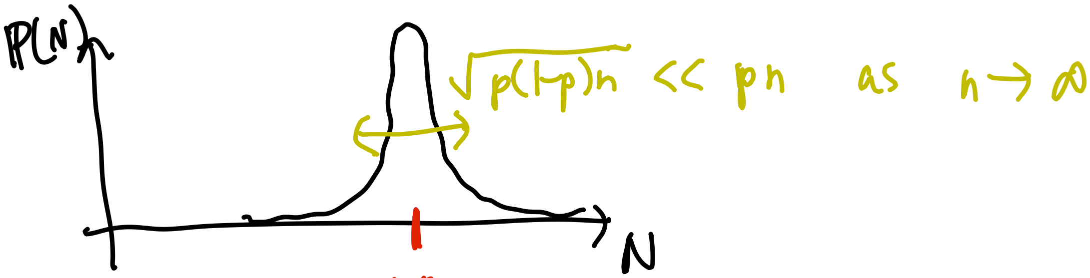
$$\mathbb{E}[N^2] = \sum_{i,j=1}^{n} \mathbb{E}[e_i e_j] = \sum_{i=1}^{n} \mathbb{E}[e_i] + 2\sum_{i<j} \underbrace{\mathbb{E}[e_i]\,\mathbb{E}[e_j]}_{\text{since } e_i, e_j \text{ independent}}$$

$e_i^2 = e_i$

$$= pn + 2 \cdot \frac{n(n-1)}{2} \cdot p^2 = (pn)^2 + np(1-p)$$

or: $\mathrm{Var}(N) = \mathbb{E}[N^2] - \mathbb{E}[N]^2 = np(1-p)$.

In other words, the distribution over errors is tightly peaked...



What are implications of this for coding?

Suppose error probability $0 < \boxed{p < 1/2}$.

↳ if $p > 1/2$, first flip all bits! $p \to 1-p \ldots$

Decode by majority vote.

$$\mathbb{P}(\text{decoder fails}) = \mathbb{P}\left(N \geq \frac{n}{2}\right) \leq \mathbb{P}\left(|N - pn| \geq \left(\tfrac{1}{2} - p\right)n\right)$$

$$\leq \frac{p(1-p)}{n\left(\frac{1}{2}-p\right)^2} \to 0 \text{ as } n \to \infty \text{ if } p < \tfrac{1}{2}.$$

because $\mathbb{P}\left(|N-pn|^2 \geq \left(\left(\tfrac{1}{2}-p\right)n\right)^2\right) \leq \dfrac{\mathbb{E}\left[(N-pn)^2\right]}{\left(\left(\frac{1}{2}-p\right)n\right)^2} = \dfrac{p(1-p)n}{\left(\left(\frac{1}{2}-p\right)n\right)^2}$

<u>Prop:</u> if $p < 1/2$, $\lim_{n \to \infty} \mathbb{P}(\text{decoding works}) = 1$.

Problem #1? What if we have faulty measurements?

$$z_i \to \theta_i \eta_i z_i \qquad \mathbb{P}(\theta_i = -1) = p_m$$

measurement error (θ_i) intrinsic error (η_i) $\qquad \mathbb{P}(\eta_i = -1) = p_e$

Call $\mathbb{P}(\theta_i \eta_i = -1) = p_e(1-p_m) + p_m(1-p_e) = p$

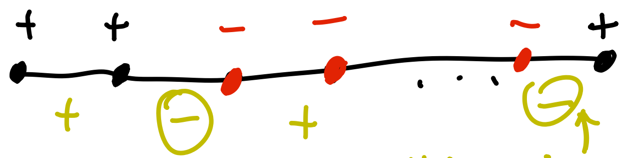
works if $p < 1/2$. $\underbrace{\text{Achieved if } p_e, p_m < 1/2.}_{\text{fault-tolerant!}}$

so not really a problem...

Problem #2: QEC measures syndromes, not qubits.

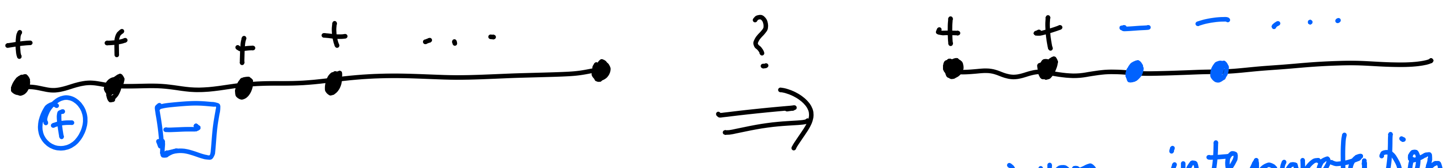
only measure $z_i z_{i+1}$ for Ising?

Danger:



only "domain wall" is flipped.

Combined w/ Problem #1 this is a real headache because:



syndrome measurement error

wrong interpretation!

MAP decoder:

given noisy $(s_1, \ldots, s_{n-1})$, what's most likely error $(z_1, \ldots, z_n)$?

$z_i = \begin{cases} 1 & \text{no error} \\ -1 & \text{error} \end{cases}$

syndrome $s_i = z_i z_{i+1} t_i$

t_i: measurement error $= \pm 1$

$$\mathbb{P}(s_1, \ldots, s_{n-1}) = \sum_{z_i} \underbrace{\mathbb{P}(s \mid z)}_{\text{conditional probability of } s_i \text{ given "known" } z_j} \mathbb{P}(z)$$

Bayes' Thm: $\mathbb{P}(s, z) = \mathbb{P}(s \mid z)\mathbb{P}(z) = \mathbb{P}(z \mid s)\mathbb{P}(s)$

$$\mathbb{P}(z \mid s) = \frac{\mathbb{P}(s \mid z)\mathbb{P}(z)}{\mathbb{P}(s)}$$

given data, want to determine z_i.

Compute: $\mathbb{P}(z) = \prod_{i=1}^{n} \begin{cases} p_e & z_i = -1 \\ 1-p_e & z_i = +1 \end{cases} \propto e^{h\sum z_i}$

where $\dfrac{e^{-h}}{e^{h}} = e^{-2h} = \dfrac{p_e}{1-p_e}$

$\mathbb{P}(s \mid z) = \prod_{i=1}^{n-1} \begin{cases} p_m & z_i z_{i+1} s_i = -1 & \text{(syndrome error)} \\ 1-p_m & z_i z_{i+1} s_i = 1 & \text{(no error)} \end{cases}$

↳ Write $e^{-2J} = \dfrac{p_m}{1-p_m}$.

So $\mathbb{P}(s|z)\mathbb{P}(z) \propto \exp\Big[\underbrace{\sum_i J s_i z_i z_{i+1} + \sum h z_i}_{\text{random-bond Ising model.}}\Big]$

Claim: finding ground state of this model $\Rightarrow$ decoder works below finite error rate

Notice that we needed global information to do the decoding. In principle it's possible to find local decoders, but this is very subtle. To see why, useful to make an analogy...

finite $p \Longleftrightarrow$ finite temperature $T>0$?

At $T>0$, 1d Ising is not a ferromagnet!

let's calculate $Z(\beta) = \sum_{z_1,\dots,z_n} e^{-\beta H} = \sum_{z_i} e^{\beta \sum z_i z_{i+1}}$

Transfer matrix strategy:

$Z(\beta) = \sum_{z_1} e^{\beta z_1 z_2} \sum_{z_2 \cdots z_n} e^{\beta(z_2 z_3 + \cdots)}$

$$z_1 = 1$$

$$Z(\beta) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix} \begin{pmatrix} Z_{2\cdots n}(z_2 = 1) \\ Z_{2\cdots n}(z_2 = -1) \end{pmatrix}$$

$$z_2 = 1$$

$$= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

eigenvector w/ eigenvalue $2\cosh\beta$

$$Z(\beta) = 2(2\cosh\beta)^{n-1}.$$

free energy $F(\beta) = -\frac{1}{\beta}\log Z(\beta) \xrightarrow{n\to\infty} -\frac{n}{\beta}\log(2\cosh\beta)$

is non-singular $\Rightarrow$ no thermodynamic transition.

More intuitive: $\Delta F_{dw} \sim \Delta E_{dw} - T\,\Delta S_{dw}$

free energy cost of domain wall

$\Delta E_{dw} \sim 1$, $\quad \Delta S_{dw} \sim \log n$

So long as $T > 0$, domain walls proliferate & destroy long-range order.