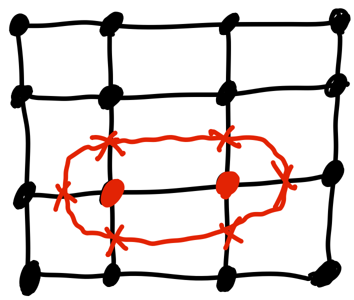
We saw that classical repetition code = Ising model, which gave us a physical picture for error correction. The real utility of the physics analogy will become clear when we talk today about the 2d Ising model:



Ising model: $H = -\sum_{i \sim j} z_i z_j$, $\quad z_i \in \{\pm 1\}$

$i \sim j$: vertex i adjacent to j

has same ground states/codewords as 1d

$+ + + +$ vs. $- - - -$

thermal Gibbs state:

$$\mathbb{P}(z) = \frac{1}{Z(\beta)} e^{-\beta H(z)}$$

z: microstate $(z_1, \dots, z_n)$

Claim: phase diagram

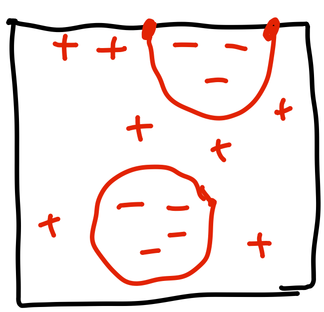
0 — ferromagnet — T_c — paramagnet $\to T = \frac{1}{\beta}$

ferromagnet: useful for error correction.

Intuition: violated checks now form closed loops!

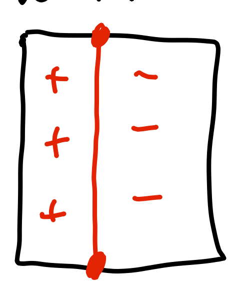
Peierls' argument: argue $\Delta F_{DW} = \Delta E_{DW} - T \Delta S_{DW} > 0$ at low T.

Claim: $\Delta E_{DW} \simeq 2(L-1)$: energy barrier separates codewords

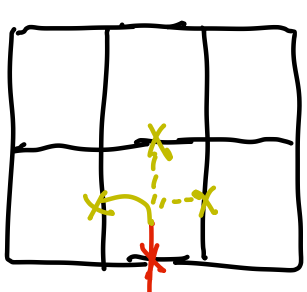


Reason: percolating spin must change $+ \to -$.

Percolation broken by long domain wall:



entropy of domain walls?



≤ 3 places to move DW after each "step"

$\Rightarrow < 3^{\ell}$ DW of length ℓ passing through edge.

So: $\Delta F_{DW} = \Delta E - T\Delta S_{DW} \sim 2\ell - T\log(3^\ell)$

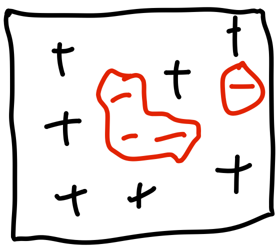
$\sim \ell(2 - T\log 3)$.

If $T < \frac{2}{\log 3}$, DW <u>unfavored</u>. This gives a lower bound on the real T_c...

Indeed there is an honest phase transition in the 2d Ising model. We won't calculate its location exactly but let's discuss the implications!

Ferromagnetism for $T < T_c$ in 2d Ising:

1. SSB / long-range order:



error clusters are small (but exist)

$$\frac{1}{Z(\beta)}\sum_{z_i} e^{-\beta H(z)} z_i z_j = \langle z_i z_j \rangle > 0 \text{ even as } \mathrm{dist}(i,j)\to\infty$$

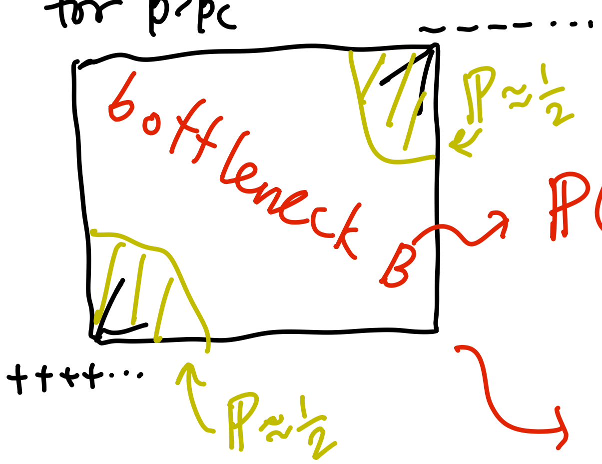
thermal avg

2. $\lim_{L\to\infty}\left[\frac{1}{L^2}F(\beta) = -\frac{1}{L^2}\frac{1}{\beta}\log Z(\beta)\right]$ has nonanalyticity at $\beta = \beta_c$

So the phase transition is thermodynamically detectable!

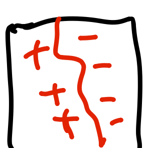
3. Measure concentration:

for $\beta > \beta_c$



overwhelmingly likely found near codeword

$\mathbb{P}(B) = \sum_{z\in B}\mathbb{P}(z) \sim e^{-\alpha L}$

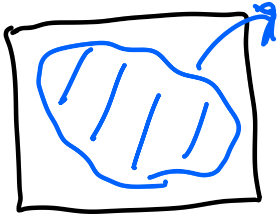


B consists of config w/ large DW

The bottleneck includes the configurations that saturate the bound on energy barrier ΔE_{DW}.

At high temperatures when we lose ferromagnetism, the absence of symmetry also has a clear signature in the Gibbs measure:

for $\beta < \beta_c$:



$\mathbb{P} \approx 1$: one large cluster in configuration space, typical states have no long-range order

④ ferromagnetism $\Rightarrow$ passive memory

local + fault-tolerant + thermal decoder!

Gibbs sampler: random process (Markov chain) that samples thermal equilibrium.

Metropolis rules: $\mathbb{P}(z \to z') \propto \min\left(1, e^{-\beta[E(z')-E(z)]}\right)$.

detailed balance: $\mathbb{P}(z \to z')\, e^{-\beta E(z)} = \mathbb{P}(z' \to z)\, e^{-\beta E(z')}$

$\Downarrow$

If we select a site x randomly from the thermal distribution,

$$\sum_z \mathbb{P}(z \to z') \cdot \frac{e^{-\beta E(z)}}{Z} = \mathbb{P}_{after}(z')$$

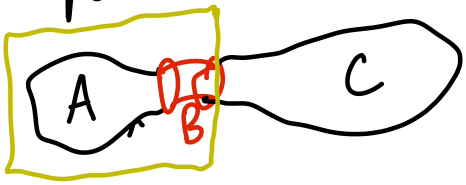
$$= \underbrace{\sum_z \mathbb{P}(z' \to z)}_{=1 \text{ (we have to go somewhere!)}} \frac{e^{-\beta E(z')}}{Z} = \mathbb{P}_{eq}(z')$$

So Gibbs sampler leaves thermal state invariant, guaranteed.

For 2d Ising model, consider Gibbs sampler:

$\mathbb{P}(z \to z') = 0$ if z & z' differ by ≥ 2 spins (single spin errors?)

Thm: expected time for Markov chain to cross bottleneck



$$\mathbb{E}[\tau] \gtrsim \frac{\min(\mathbb{P}_{eq}(A), \mathbb{P}_{eq}(C))}{\mathbb{P}_{eq}(B)}$$

[if we draw initial site from $\mathbb{P}_{eq}$]

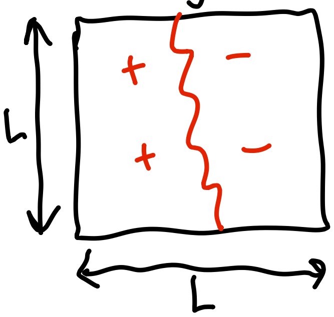
Proof sketch: If dynamics restricted to A & B, $\mathbb{P}_{AB}(B) = \dfrac{\mathbb{P}_{eq}(B)}{\mathbb{P}_{eq}(B)+\mathbb{P}_{eq}(A)}$

$$\Rightarrow \sum_{t=0}^{\tau} \mathbb{P}[\text{visit } B \text{ at time } t) = (1+\tau)\mathbb{P}_{AB}(B)$$

Restricted dynamics = full dynamics until we hit B

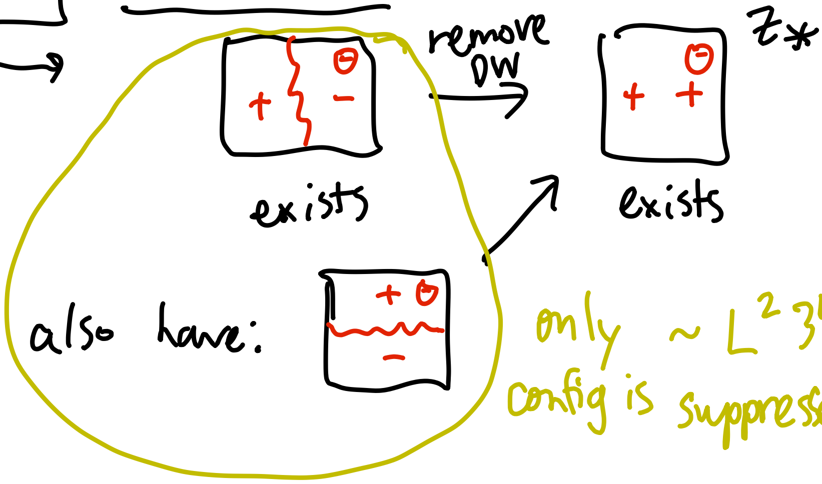
↳ $1 \sim \mathbb{E}[\tau]\mathbb{P}_{AB}(B)$ needed to be likely to hit B.

2d Ising model has low-T bottleneck: $B = \{z$ w/ DW of length $L\}$



Thm: $\mathbb{P}_{eq}(\text{bottleneck}) \sim e^{-\alpha L} \Rightarrow \tau \gtrsim e^{\alpha L}$

Proof sketch: Peierls!

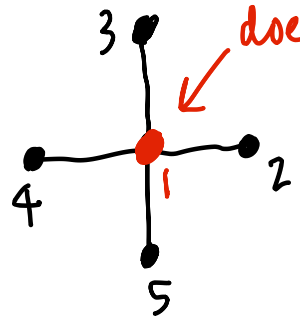


only $\sim L^2 3^L$ such large DWs, each config is suppressed relative to z_* by $e^{-\beta L}$

We only should remove the long DW to leave bottleneck.
Indeed a typical thermal config has finite density of small DWs.
Don't expect to remove all of the errors all at once!

Now let's see why Gibbs sampler ⇒ local, fault-tolerant decoder.

One step of Gibbs sampler:



does flipping lower E ?

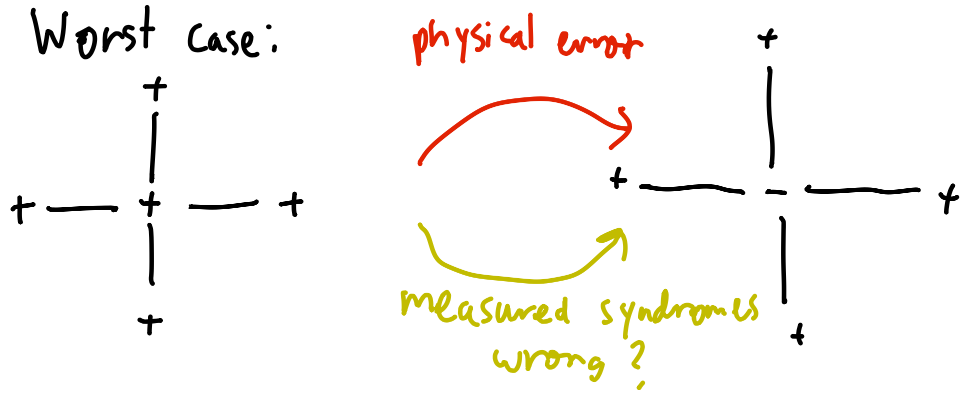
$$\Delta E = -(-z_1)(z_2+z_3+z_4+z_5) - (-1)z_1(z_2+z_3+z_4+z_5)$$
$$= \underbrace{2(z_1z_2+z_1z_3+z_1z_4+z_1z_5)}_{\text{only needs local information about syndromes}}$$

Safe for QEC!

Apply bit flip to 1 w/ probability $\min(1, e^{-\beta \Delta E})$

Why does this imply there's a fault-tolerant decoder?

Worst case:



rates:

$$\gamma_{phys} + \gamma_{dec\ fail} = e^{-8\beta}\gamma_{dec\ success}$$

So the relative rates of physical error per bit + failed decoding per bit can be O(1) vs. the decoding rate and still satisfy this equation! Guaranteed a threshold!

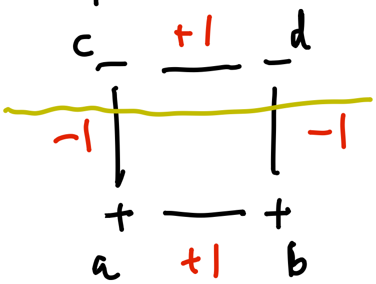
For other error patterns? $\gamma_{phys} + \gamma_{dec\ fail} + \gamma_{add\ error} = e^{-O(\beta)}\gamma_{dec\ suc}$

if we want to match the perfect Gibbs sampler!

The upshot of this argument is that we see that a physical phase of matter can inspire a robust fault-tolerant scheme for error correction. Indeed, classical info tech (magnetic memory) was based on our ability to safely store a bit in a magnet...

Why did 2d Ising work so much better than 1d?

Energy in 2d contained <u>redundant</u> checks:



product $(Z_a Z_b)(Z_b Z_c)(Z_c Z_d)(Z_d Z_a) = +1$

DWs formed closed loops

This provided us with enough redundancy to get a low-T phase w/ passive memory!