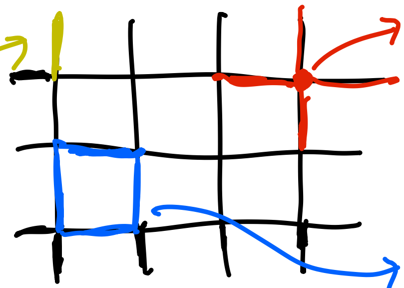
We're now ready to introduce a quantum analogue of 1d Ising, which will be a workhorse for our study of quantum codes:

Toric code:

qubit on each edge

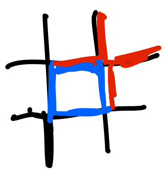


periodic boundary conditions

at each vertex v:
X-check: $\prod_{e \in \partial v} X_e =: X^{\partial v}$
(edge adj to v)

around each face f:
Z-check: $\prod_{e \in \partial f} Z_e =: Z^{\partial f}$

Each X_v & Z_f share either 0 or 2 edges:



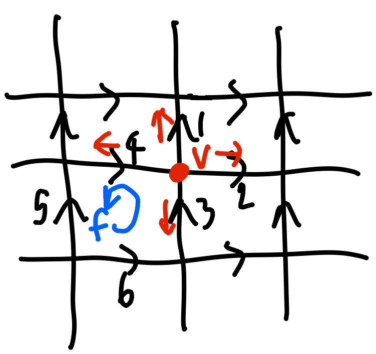
$$\Rightarrow \quad [X^{\partial v}, Z^{\partial f}] = 0 \quad \forall\, v, f$$

Key advantages of toric code:

- geometric locality in 2d $\Rightarrow$ nice for experiment? (esp. superconducting circuit)
  we'll fix bc's later...
- scalable $d \sim \sqrt{n}$ which we'll prove later.

Before we dive in to QEC w/ 2d codes, we'll spend a bit of time understanding where it comes from, and analogies to physics...

Observe an analogy with (discrete) vector calculus:



function: G_v defined on vertices $(\mathbb{R}^V)$
(0-form)

vector field: u_e defined on edges $(\mathbb{R}^E)$
(1-form)

dual function: ϕ_f defined on faces. $(\mathbb{R}^F)$
(2-form)

(d_0)
grad: $\mathbb{R}^V \to \mathbb{R}^E$ $\quad (\vec{u} = \nabla G)$

(d_1)
curl: $\mathbb{R}^E \to \mathbb{R}^F$ $\quad (\phi = \partial_x u_y - \partial_y u_x)$

For any G, $\nabla \times (\nabla G) = 0. \Rightarrow$ curl $\cdot$ grad $= 0$ as operators

$\Rightarrow d_1 d_0 = 0.$

What do d_0 & d_1 act like as matrices? Using diagram above,

$\underbrace{\langle e_1 | d_0 | v \rangle = 1 = \langle e_2 | d_0 | v \rangle}_{\text{edges oriented away from } v}$ $\quad$ $\underbrace{\langle e_3 | d_0 | v \rangle = \langle e_4 | d_0 | v \rangle}_{\text{edges oriented into } v.} = -1$

$\underbrace{\langle f | d_1 | e_3 \rangle = \langle f | d_1 | e_6 \rangle = 1}_{\text{face "circulation" aligned w/ edge}}$ $\quad$ $\underbrace{\langle f | d_1 | e_4 \rangle = \langle f | d_1 | e_5 \rangle = -1}_{\text{face anti-aligned}}$

These integer valued matrices can be understood on $\mathbb{F}_2$, where:

on $\mathbb{F}_2$: $\langle e | d_0 | v \rangle = \begin{cases} 1 & e \text{ adjacent to } v \\ 0 & \text{else} \end{cases}$

$\langle f | d_1 | e \rangle = \begin{cases} 1 & f \text{ adjacent to } e \\ 0 & \text{else} \end{cases}$

$|v\rangle \longrightarrow$ X-checks $\quad$ $|e\rangle \longrightarrow$ qubits $\quad$ $|f\rangle \rightarrow$ Z-checks

$\Rightarrow \quad d_0 = H_X^T$ and $d_1 = H_Z$. $\quad H_Z H_X^T = d_1 d_0 = 0$

So the mathematics of curl & grad directly leads to a choice of parity-check matrices! Even better, they tell us what the logical operators are!

Recall: logical X parameterized by $\mathcal{L}_X = \dfrac{\ker(H_Z)}{\operatorname{im}(H_X^T)} = \dfrac{\ker(d_1)}{\operatorname{im}(d_0)}$

This is a famous object in mathematics, let's quickly explain!

Def: cochain complex on $\mathbb{F}_2$: sequence of vector spaces

$V_2 \xleftarrow{d_1} V_1 \xleftarrow{d_0} V_0$; $\quad d_0 : V_0 \to V_1$; $d_1 : V_1 \to V_2$

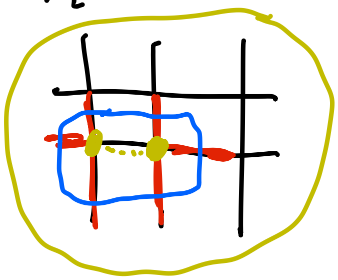
maps btwn them obey $d_1 d_0 = 0$.

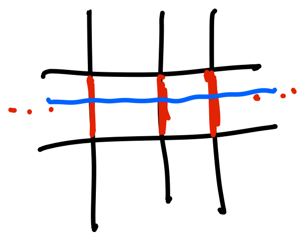
Def: Cohomology group: $H^1 = \ker(d_1)/\mathrm{im}(d_0)$

$\ker(d_1)$: vector fields with "no curl"

$\Rightarrow \mathbb{F}_2$: subset of edges touching any face even number of times.

e.g.





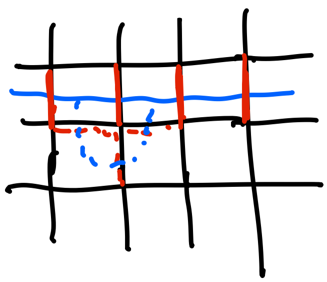
$\mathrm{im}(d_0)$: vector fields that are gradients

Useful to connect included edges on dual lattice:

loop should be <u>closed</u>.

Recall: $\ker(d_1)/\mathrm{im}(d_0) \Rightarrow x_1 \sim x_2^{\in \ker(d_1)}$ if $x_1 = x_2 + \underbrace{d_0 G}_{}$

differ by gradient



$\rightsquigarrow$ these two elements are <u>cohomologous</u>

(i.e. in same equivalence class)

H^1 counts the number of <u>distinct</u> cohomology classes

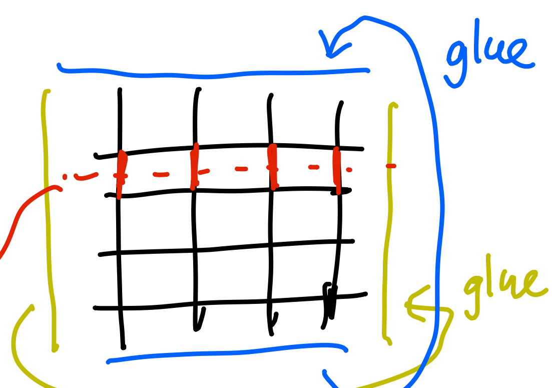
$\Rightarrow$ non-stabilizer equivalent <u>X-logicals</u>.

Thm: If square lattice has periodic BCs (2d torus),

$H^1 = \mathbb{F}_2^2$ (i.e. $k = 2$)

Rather than give a formal proof,

we sketch why it must be so:



closed (dual) loop that's <u>non-contractible</u>

Earlier we saw that contractible loops come from stabilizers, so only horizontal/vertical non-contractible loops (and their sum) are logicals.

on $L \times L$ lattice of vertices, have $n = 2L^2$ qubits

code distance $d_X = L = \sqrt{n/2}$. (clear from argument above)

There's a similar story for Z-logicals we'll quickly go through.

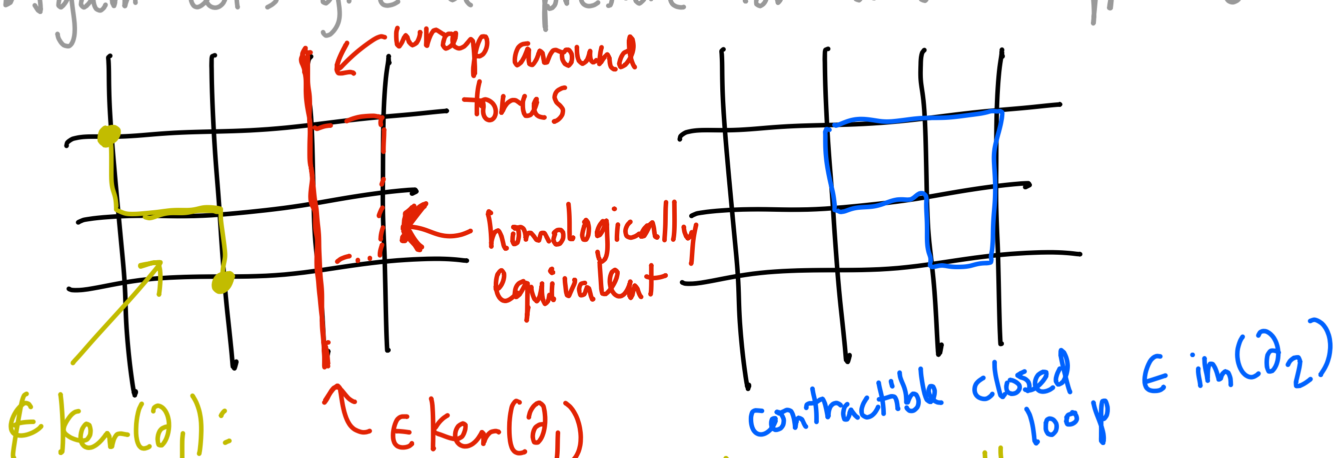
Def: <u>chain complex</u> on $\mathbb{F}_2$ from reading cochain complex in reverse

$$V_2 \xrightarrow{\partial_2} V_1 \xrightarrow{\partial_1} V_0 \quad \text{where} \quad \partial_{n+1} = (d_n)^T$$

Now $\partial_n \partial_{n+1} = 0$ so we can define

Def: <u>homology group</u> $H_1 = \ker(\partial_1)/\mathrm{im}(\partial_2)$.

Again let's give a picture for what's happening:



open string has endpoints $\Rightarrow$ syndrome $s_X = H_1 e_Z$ ← Z-error

$$\mathcal{L}_Z = H_1 = \frac{\{\text{closed loops}\}}{\{\text{bdy of contractible loops}\}}$$

We've already proved the following in a previous lecture!

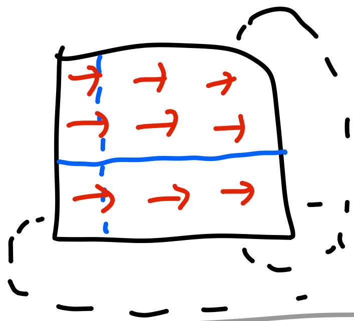
Thm: $\dim(H^1) = \dim(H_1)$ i.e. # of X-logicals = # of Z-log.

There's now a very nice pictorial interpretation for this result in terms of topology!

H_1: non-contractible cycles on manifold

H^1: curl-free vector field, $\neq$ gradient:

on torus:
(on $\mathbb{R}$ not $\mathbb{F}_2$)



uniform $\vec{v} \neq \nabla x$ since x not global coordinate!

With this detour into topology, let's recap the implication for quantum codes.

$$\begin{array}{ccccc} \text{(faces)} & (H_Z) & \text{(edges)} & (H_X^T) & \text{(vertices)} \\ \mathbb{F}_2^{m_Z} & \underset{\partial_2}{\overset{d_1}{\rightleftarrows}} & \mathbb{F}_2^{n} & \underset{\partial_1}{\overset{d_0}{\rightleftarrows}} & \mathbb{F}_2^{m_X} \\ \text{(Z-checks)} & & \text{(qubits)} & & \text{(X-checks)} \end{array}$$

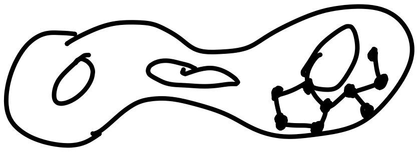
(cochain)
(chain)

X-logicals: H^1 Z-logicals: H_1 $\Rightarrow$ $k=2$ on torus

canonical pairing from (co)homology!

distance = length of smallest non-contractible cycle $\sim \sqrt{n}$.

Build quantum code on <u>any tiling</u> of 2d surface:



$\rightarrow$ chain complex $\Rightarrow$ H_X & H_Z

X-check on vertex Z-check on face

<u>Thm</u>: on <u>genus-g</u> surface Σ_g (w/ g handles):

$$H_1(\Sigma_g) = H^1(\Sigma_g) = \mathbb{F}_2^{2g} \quad \Rightarrow \quad k = 2g$$

$\Rightarrow$ topological code

It's harder in experiment to make surfaces w/ holes so soon we'll come up w/ a smarter way to store $k \gg 1$...

One more key thing (for later theory of fault-tolerance) is that the toric code is a quantum LDPC code:

Def: <u>quantum LDPC code</u>: H_X & H_Z are both LDPC

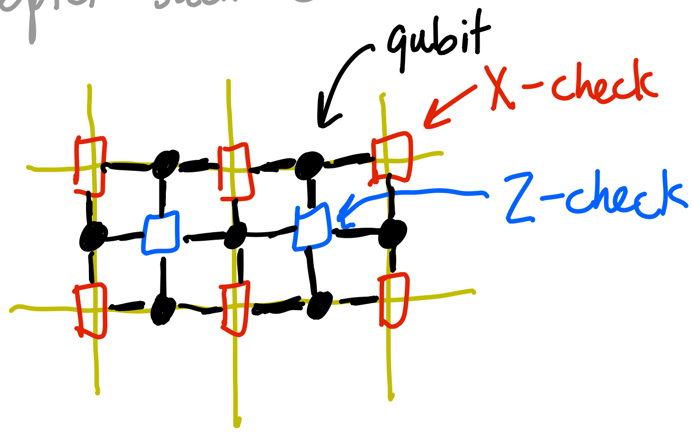
[each check has $\leq \Delta_C$ bits] [each bit in $\leq \Delta_B$ checks]

This definition is meaningful as $n \to \infty$ for fixed Δ_B, Δ_C.

Toric code has $\Delta_B = \Delta_C = 4$.

Often useful to depict such codes via Tanner graph:

for toric code:



$H_X H_Z^T = 0 \Rightarrow$ every $(\square, \square)$ pair shares even # of adjacent qubits

Finding Tanner graphs (for useful codes w/ good n, k, d) subject to this constraint & LDPC condition is <u>hard</u>. Why the topological analogy we discussed today is so important.