PHYS 7810: Quantum Error Correction **Fall 2026**

Homework 1

Due: September 8 at 11:59 PM. Submit on Canvas.

Problem 1 (Spontaneous emission): Consider an atomic qubit where $|0\rangle$ is a ground state, while $|1\rangle$ is an excited state. These states are chosen such that $|1\rangle$ may decay via spontaneous emission of a photon to $|0\rangle$.

15 **A:** A "microscopic" account of this decay process, using quantum channels, proceeds as follows. Let $|\circ\rangle$ denote the no-photon state while $|\gamma\rangle$ denotes the photon state. Assuming that there is no photon to start with, the state of the system after some time becomes:

$$U\left[\alpha|0\circ\rangle + \beta|1\circ\rangle\right] = \alpha|0\circ\rangle + \beta\left[\sqrt{1-p}|1\circ\rangle + \sqrt{p}|0\gamma\rangle\right] \tag{1}$$

where p is the probability of a decay event, and α, β are generic wave function coefficients. We will prefer to work in the language of open quantum systems, where we trace out the photon states. Show by explicit calculation that after tracing out the photon,

$$\operatorname*{tr}_{\text{photon}}\left[U\rho\otimes|\circ\rangle\langle\circ|U^\dagger\right] =: \mathcal{N}(\rho) =: K_1\rho K_1^\dagger + K_2\rho K_2^\dagger \tag{2}$$

where $\rho = |\psi\rangle\langle\psi|$, with $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, is the initially pure state of the atomic qubit alone, and the Kraus operators are

$$K_1 = |0\rangle\langle 0| + \sqrt{1-p}|1\rangle\langle 1|, \tag{3a}$$
$$K_2 = \sqrt{p}|0\rangle\langle 1|. \tag{3b}$$

Verify that $\mathcal{N}$ is a valid quantum channel.

10 **B:** Suppose we used these atomic qubits to build one of the three codes described in Lecture 2. Which, if any of these codes, can correct for a single spontaneous emission error on any of the qubits? Assuming you could easily measure any product of Pauli matrices, explain how you would perform error correction for this system. Does the error correction scheme depend on p?

Problem 2 (Storing a logical qubit inside a single qudit): Suppose that we have an experimental system with access to a single physical qudit with q internal levels, i.e. the Hilbert space is

$$\mathbb{C}^q = \mathrm{span}\{|0\rangle, |1\rangle, \ldots, |q-1\rangle\}. \tag{4}$$

There is a natural generalization of Pauli X and Z operators on qubits:

$$X|j\rangle = |j+1 \pmod{q}\rangle, \tag{5a}$$
$$Z|j\rangle = \mathrm{e}^{2\pi \mathrm{i} j/q}|j\rangle. \tag{5b}$$

A short calculation shows that X and Z are unitary matrices, obeying $X^q = Z^q = I$ and

$$XZ = ZX\mathrm{e}^{-2\pi\mathrm{i}/q}. \tag{6}$$

This setup arises in the context of bosonic GKP codes, which can be realized in superconducting circuits.

There are beautiful codes, which I call "torus codes", that allow you to store logical qubits inside the larger physical qudit. In this problem we will look at two such codes. The first such code uses $q = 8$, and imposes stabilizers $S_1 = Z^4$ and $S_2 = X^4$.

10 **A:** Check that $[S_1, S_2] = 0$. Show that the codespace is two-dimensional with logical codewords

$$|0_\mathrm{L}\rangle = \frac{|0\rangle + |4\rangle}{\sqrt{2}}, \tag{7a}$$
$$|1_\mathrm{L}\rangle = \frac{|2\rangle + |6\rangle}{\sqrt{2}}. \tag{7b}$$

Let P be the projector onto $\mathrm{span}\{|0_\mathrm{L}\rangle, |1_\mathrm{L}\rangle\}$. Find a set of logical Pauli operators X_L and Z_L, obeying $X_\mathrm{L}^2 P = Z_\mathrm{L}^2 P = P$ and $X_\mathrm{L} Z_\mathrm{L} P = -Z_\mathrm{L} X_\mathrm{L} P$, as in Lecture 2. Show that $[X_\mathrm{L}, S_1] = [X_\mathrm{L}, S_2] = [Z_\mathrm{L}, S_1] = [Z_\mathrm{L}, S_2] = 0$.

10 **B:** Show that one set of correctable errors is $\mathrm{span}\{I, X, Z, XZ\}$. Is this choice unique? Why or why not?

10 **C:** Explicitly write down the recovery channel $\mathcal{R}$, following Lecture 3. Give a clear physical explanation for what it does and how you could implement it, in principle, experimentally.

5 **D:** Show that X and X^{-1} cannot both be corrected, and explain why this is so.

Now, let's discover a code that can correct all of the "smallest errors" in the system: namely, we'd like the correctable error set to be exactly $\mathrm{span}(I, X, X^{-1}, Z, Z^{-1}) = \mathrm{span}(E_a)$. We also do not want any of these errors to be degenerate: following Lecture 3, we demand $\alpha_{ab} = \delta_{ab}$.

5 **E:** Prove that you need $q \geq 10$ for this to be possible.[1]

10 **F:** It is possible to find such a code using $q = 10$, with a single stabilizer of the form $S = X^a Z^b$, which has the property that $S^5 = I$, and the eigenvalues of S are the five roots of unity $\mathrm{e}^{2\pi\mathrm{i}j/5}$ for $j = 0, \ldots, 4$. Find a valid choice of a and b, being sure to explain why the choice works. As part of your answer, you should explain conceptually how to decode this code, although you don't need to explicitly write down $\mathcal{R}$.[2] Find logical operators for this code.

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[1] *Hint:* Think about the P_a introduced in the proof of the Knill-Laflamme conditions. Under the conditions we are demanding, you should find that each detectable syndrome must have at least a two-dimensional subspace.

[2] *Hint:* Each independent decodable error should be identified with a distinct eigenvalue of S.

Problem 3 (Lower bound on the threshold for 1d Ising): Consider the classical 1d Ising model/repetition code from Lecture 4. Suppose that we start in the codeword $+1$, and then an arbitrary fraction p_e of all bits are flipped to -1. Let $z_i \in \{\pm 1\}$ be the resulting values of the bits after these errors. Do not assume that the errors act randomly – they may be correlated with each other in an "adversarial" way to try and fool a decoder!

Now, as we would need to do in a quantum code, we can only measure the syndromes $s_i = z_i z_{i+1}$ for $i = 1, \ldots, n-1$. Yet, the actual measured syndrome is $s_i = z_i z_{i+1} t_i$, where $t_i \in \{\pm 1\}$ accounts for the possibility that the syndrome was measured incorrectly. Assume that a fraction p_m of all $t_i = -1$; again, do not assume the incorrectly measured syndromes are random!

A simple strategy for decoding is to determine the most likely error, following the maximum likelihood estimation procedure described in Lecture 4, where we saw that the most likely error $\widehat{z}_i$ corresponds to the ground state of a 1d random bond Ising model. (Of course, this mapping does assume the errors are random!) One decoding strategy is then to simply apply this most likely error $\widehat{z}_i$, and ask whether

$$\frac{1}{n}\sum_{i=1}^{n} \widehat{z}_i z_i > 0? \tag{8}$$

If so, we're closer to the initial codeword than its logical counterpart, so we succeeded.

15 **A:** Show that when p_e and p_m are both sufficiently small, the decoder is guaranteed to work – i.e. (8) is obeyed. To do this, compare the highest possible energy of the state $\widehat{z}_i = z_i$ with error fractions p_e and p_m, with the lowest possible energy of any state that violates (8). Show that, in the $n \to \infty$ limit, the state with $\widehat{z}_i = z_i$ necessarily has lower energy whenever

$$\left(\frac{1}{2} - 2p_\mathrm{e}\right) \log \frac{1-p_\mathrm{e}}{p_\mathrm{e}} > p_\mathrm{m} \log \frac{1-p_\mathrm{m}}{p_\mathrm{m}}. \tag{9}$$

Deduce that the decoder necessarily works for *random* errors with $p_\mathrm{e} = p_\mathrm{m} = p$ so long as

$$p \leq \frac{1}{6}. \tag{10}$$

10 **B:** Do you think we have a high probability of exactly correcting all the errors in the limit $n \to \infty$, i.e. finding $\widehat{z}_i = z_i$? Why or why not?