PHYS 7810: Quantum Error Correction **Fall 2026**

Homework 2

Due: September 22 at 11:59 PM. Submit on Canvas.

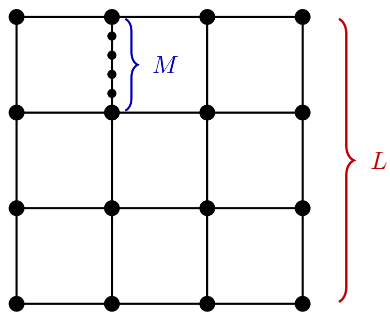

Figure 1: Sketch of the modified 2d Ising model where every edge of the square lattice is replaced by M sequential edges.

Problem 1 (A bad 2d Ising model): Consider the Ising model (repetition code) on the following modified $L \times L$ 2d square lattice, where each edge is replaced by M sequential edges (i.e. $M-1$ additional vertices are added along each edge): see Figure 1. When $M, L \gg 1$, the total number of bits is then $n \approx 2ML^2$. Freely use $M, L \gg 1$ to simplify answers as you wish.

10 **A:** What is the energy barrier that separates the two logical codewords of this peculiar repetition code? The energy of a configuration is given by the natural generalization of the formula from Lectures 4 and 5: we check whether each pair of two adjacent bits are the same or not.

15 **B:** Fix $M \gg 1$ and then take $L \to \infty$. Following Lecture 5, estimate the critical temperature T_c, such that there is ferromagnetism and the code is a passive memory for $T < T_\mathrm{c}$.

5 **C:** Take $M \sim L^\alpha$ for some exponent $\alpha > 0$. Deduce that, in the thermodynamic limit, diverging energy barriers are not sufficient for passive memory.

Problem 2 (Classical and quantum Hamming codes): In Lecture 6, we described the classical 7-bit Hamming code. There is a natural generalization of this code given $n = 2^r - 1$ physical bits for any $r \geq 3$: take the columns of the parity-check matrix H to be all possible non-zero vectors in $\mathbb{F}_2^r$, each included exactly once.

10 **A:** Find the values of k and d for this classical Hamming code.

10 **B:** Show that you can make a quantum CSS code where $H_X = H_Z = H$ (with H the classical parity-check matrix of the Hamming code); this is called a quantum Hamming code.

15 **C:** Find the values of k and d for this quantum Hamming code.

Problem 3 (Quantum Hamming bound): Consider a general quantum Pauli stabilizer code.

10 **A:** Suppose that we want a code with a distance $d = 2t+1$; namely, the code must correct every error on $\leq t$ bits. Find a lower bound on the number of unique errors we must be able to distinguish, assuming that every Pauli error of weight $\leq t$ maps to a unique syndrome, and use this to deduce a lower bound on n as a function of k and t. Your result will be the quantum Hamming bound – the generalization of the classical Hamming bound from Lecture 6 to quantum codes.

20 **B:** Show that if $t = k = 1$, your bound from **A** gives $n \geq 5$. An explicit example of such a code with $n = 5$ has the four stabilizers

$$S_1 = X_1 Z_2 Z_3 X_4, \tag{1a}$$
$$S_2 = X_2 Z_3 Z_4 X_5, \tag{1b}$$
$$S_3 = X_1 X_3 Z_4 Z_5, \tag{1c}$$
$$S_4 = Z_1 X_2 X_4 Z_5. \tag{1d}$$

Write the 4×10 parity-check matrix H over $\mathbb{F}_2$ for this code. Then, find (one choice of) smallest-weight logical X and Z operators for this code. Explain how you can use the measured syndrome to correct arbitrary single-qubit errors on any physical qubit: in particular, provide a table showing which syndrome corresponds to which correctable single-qubit error.

5 **C:** Suppose we want codes that, as $n \to \infty$, have a fixed ratio $r = k/n$. We'd like the ratio $d/n = a$ to be as large as possible. Find an upper bound on a using the result of **A**.