PHYS 7810: Quantum Error Correction — Fall 2026

Homework 3

Due: October 13 at 11:59 PM. Submit on Canvas.

15 **Problem 1 (Wilson loops at finite temperature):** In high energy physics, one often uses **Wilson loop** observables to analyze lattice gauge theories, like the toric and surface code. In this problem, consider the surface code on an $L \times L$ square lattice. Take a square loop γ of side length $b \ll L$ in the interior of the *unrotated* surface code (e.g. as we first constructed in Lecture 11 using the ghost vertices), and consider the Wilson loop observable

$$Z^{\gamma} := \prod_{e \in \gamma} Z_e. \tag{1}$$

Let ρ_β denote the thermal density matrix for the surface code.

Show that

$$\mathrm{tr}\left(\rho_\beta Z^{\gamma}\right) = \mathrm{e}^{-\alpha \cdot b^2} \tag{2}$$

and find the constant α as a function of β.[1] It turns out that the exponential decay of the Wilson loop in terms of volume b^2, rather than perimeter $\sim 4b$, is a gauge-theoretic way of seeing that the topological phase in the 2d surface code is destroyed by thermal fluctuations.

Problem 2 (Rydberg atom entangling gate): In Rydberg atom arrays, the most natural entangling gate corresponds to a CZ gate. Between qubits i and j, the CZ gate is defined as:

$$\mathrm{CZ}_{ij} = \mathbb{I} - 2|1\rangle\langle 1|_i \otimes |1\rangle\langle 1|_j. \tag{3}$$

15 **A:** Show that CZ_{ij} is a Clifford gate. Show explicitly how CZ_{ij} transforms all single-qubit Pauli X and Z operators. In addition and equivalently, write the 4×4 symplectic $\mathbb{F}_2$ matrix corresponding to CZ_{12} acting on $n = 2$ qubits.

10 **B:** We will want to understand how to prepare the GHZ state

$$|\mathrm{GHZ}\rangle := \frac{|000\cdots\rangle + |111\cdots\rangle}{\sqrt{2}} \tag{4}$$

with Rydberg atoms. Find an explicit set of n commuting stabilizers that stabilize $|\mathrm{GHZ}\rangle$.

10 **C:** Using only experimentally accessible single-qubit rotation gates (H and S on any one qubit), and CZ gates, find a circuit that prepare the GHZ state, starting from the initial state $|00\cdots\rangle$. Explain how to use the stabilizer formalism in your analysis to track the evolution of the state through the circuit.

[1] *Hint:* Use the fact that the parity-checks of the surface code are non-redundant to vastly simplify the calculation!

Problem 3 (Holes in the surface code): One mechanism for enhancing the surface code to store more logical qubits, while maintaining a simple planar architecture with only spatially local interactions, is to add holes to the surface code.

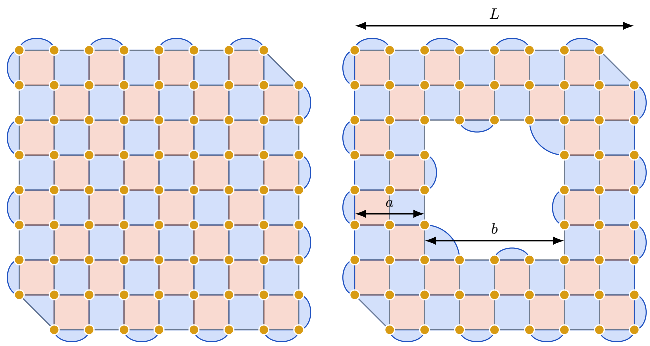

Figure 1: *Left:* A surface code with only rough boundaries. *Right:* A surface code with a hole in the middle. We use only rough boundaries for both the interior and exterior boundary. Recall that red denotes X-checks and blue denotes Z-checks, as in the lecture notes.

10 **A:** Consider the left surface code in Figure 1 which only has rough boundaries. What is n, the number of physical qubits? What are $\mathrm{rank}(H_X)$ and $\mathrm{rank}(H_Z)$, and therefore k?[2] Contrast with the rough surface code discussed in Lecture 11 and explain the difference succinctly.

15 **B:** Now consider the right surface code in Figure 1, which has a hole in the middle. Again calculate n and k. What are the code distances d_X and d_Z, and where are the (smallest) logical representatives? Be sure to give a physical explanation for why the logicals are where they are, so that you could easily explain how your answer generalizes to a surface code (with hole) of any size.

10 **C:** Suppose that we changed the top and bottom boundaries to be smooth, instead of rough (recall the definitions in Lecture 11). Again, calculate n, k, d_X and d_Z. You should find that the answer differs from **B** – explain why by giving a clear physical picture for where all of the logical operators are.

10 **D:** Return to the scenario of **B**, where all interior and exterior boundaries are rough. Now, however, consider a surface code consisting of

$$L = (m+1)a + m(b-2) \tag{5}$$

Now place m^2 equally spaced holes in the surface code, with the distance between each hole equal to a, and the width of each hole equal to b. The case of $m = 1$ is given in the right half of Figure 1. Deduce the values of n, k, d_X and d_Z for this code. Given your earlier work, a clever argument will suffice – you do not need to justify it in full detail.

5 **E:** Argue using the BPT bound (Lecture 11) that there cannot be a qualitatively better way to encode extra logical qubits in a patch of surface code – namely, if you wish to maintain a given distance $d = \min(d_X, d_Z)$, you can't encode too many more logical qubits k than you did in the construction of **D**.

[2] *Hint:* You should *not* do this by brute force. Instead, try to follow arguments we made in Lectures 9-11 to deduce how many of the X-checks and Z-checks are independent.