

Homework 5

Due: November 10 at 11:59 PM. Submit on Canvas.

Problem 1 (From the Bacon-Shor code to the surface code): In Lecture 19, we discussed the 9-qubit Bacon-Shor code. There is a natural generalization on an $L \times L$ square lattice. Label the qubits by the coordinates (i, j) of a qubit on the lattice. The Bacon-Shor code is a subsystem CSS code with gauge group

$$\mathcal{G} := \langle X_{i,j} X_{i,j+1}, Z_{i,j} Z_{i+1,j} \rangle. \quad (1)$$

The case of $L = 3$ is the 9-qubit Bacon-Shor code we discussed in Lecture 19.

- 15 **A:** Generalize Lecture 19 and find the center $Z(\mathcal{G})$ and the stabilizer group $\mathcal{S}$.
- 10 **B:** Suppose that each qubit undergoes a Pauli Y error independently with probability p . Show that the probability that no undetectable error has occurred goes to 0 as $L \rightarrow \infty$. To do this, show that with high probability, there exists (at least) one site in every row with an error. Show why this leads to an undetectable error, even assuming zero errors in the (gauge) syndrome measurement process.
- 10 **C:** Show how to gauge fix the Bacon-Shor code to find the (rotated) surface code from Lecture 11, as an ordinary CSS code with stabilizer group $\mathcal{S}_{\text{surf}}$. Verify that

$$\mathcal{S} \subset \mathcal{S}_{\text{surf}} \subset \mathcal{G}. \quad (2)$$

- 10 **D:** Why does the surface code have a threshold under our simple error model above, when the Bacon-Shor code does not?

Problem 2 (Logical operators in color codes): In this problem, we will describe the nature of logical operators in color codes, and further illustrate how to choose boundary conditions (analogous to rough surface codes in Lecture 11) so that color codes have distances that grow with the system size.

- 15 **A:** Begin with a 2-colex cellulation of a two-dimensional sphere, following Lecture 15. Explain the following stabilizer redundancies for X -checks, and why we will see an analogous redundancy for Z -checks:

$$\prod_{\text{red } f} X^f = \prod_{\text{blue } f} X^f = \prod_{\text{yellow } f} X^f. \quad (3)$$

Now, apply Euler's formula (Lecture 11), which on a sphere reads: $V - E + F = 2$. Why must V be even? Relate E , V and F to the number of X/Z checks, and of qubits – not necessarily in that order! Use, without proof, the fact that (3) are the only check redundancies, to show $k = 0$.

- 10 **B:** Now remove a single qubit along with the 3 faces it is adjacent to. Show that the new code will have $k = 1$. However, show that there exists a logical operator of the code located near the removed qubit. If each face has at most M vertices in it, show that $d \leq M - 1$.

One way, then, to think about making a color code with a good distance is to look for a funny triangulation where the three faces (around the removable vertex) each contain a large $O(L)$ number of vertices, if the 2-colex contains $O(L^2)$ vertices more generally. A concrete such construction is shown in Figure 1.

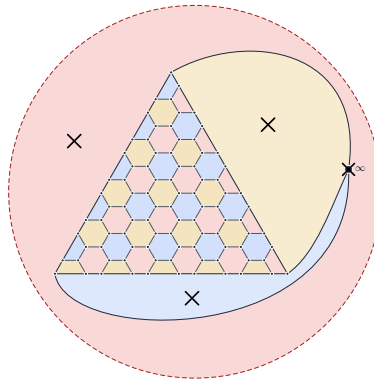


Figure 1: A triangulation of a two-dimensional sphere (imagine the large red disk wraps around behind the page); remove the vertex at ∞ to create a nicer triangular 3d color code in the plane.

- 15 **C:** There is an elegant interpretation, generalizing the notions of “rough” and “smooth” boundaries from the surface code, for what Figure 1 has done. Modify what we did in Lectures 20 and 21 for the 3d gauge color code, and argue that there are three types of excitations in a 2d color code, associated with red/blue/yellow error strings. Describe the rules for how these particles can merge or annihilate each other, analogous to what we did for e/m particles in Lecture 10. Deduce that the role of each colored boundary in Figure 1 is to provide a boundary on which a certain kind of error string can terminate. Explain intuitively, in this description, what a general logical operator is.¹ How will the code distance scale with system size?
- 15 **D:** Propose the generalization of this story for the 3d gauge color code. You may wish to think of replacing the triangle 2-colex in Figure 1 with a tetrahedron 3-colex. The details of the cellulation are unimportant here, and you don't need to discuss them. Instead, explain how a boundary would be truncated in the 3-colex at the faces of your tetrahedron. Then, give a strong argument (a proof is not needed) for the code distance of the resulting code, explaining where the logicals act.

¹Hint: The logical operators found a la **B** may be a bit subtle; make sure your explanation incorporates them!