

Homework 6

Due: December 1 at 11:59 PM. Submit on Canvas.

Problem 1 (Passive decoder for the 4d toric code): We wish to implement the Gibbs sampler at inverse temperature $\beta = 1.0$ for the 4d toric code described in Lecture 22.

- 15 **A:** Explain why the transition rates of a Gibbs sampler at this β can form the basis for a self-correcting quantum memory. Using the Metropolis sampling rules and using the parity-checks of the 4d toric code, what is the smallest physical qubit error probability p_e that can be tolerated to ensure that the decoder can match the transition rates of this Gibbs sampler? Assume that $p_m = 0$. Compare your answer to the thresholds for the 2d surface code.
- 10 **B:** Argue that we can use MFQEC to implement a Gibbs sampler as a local, scalable decoder for the 4d toric code, and that this decoder will have a finite MFQEC circuit depth and a non-zero threshold as we scale the code to be larger. You do not need to be rigorous but sketch how the argument would go, drawing upon earlier results as needed.

Problem 2 (Greedy decoder): Consider a random (Δ_B, Δ_C) classical LDPC code. A greedy decoder works by looking at a given syndrome $s = Hx$ and asking whether there is any single bit to flip that reduces the size of the syndrome. In equations: is there a y with $|y| = 1$ such that $|H(x + y)| < |Hx|$? If so, then update $x \rightarrow x + y$ and iterate this procedure until we either reach $s = 0$ or a local minimum in the “energy landscape”.

- 20 **A:** Using the proof of linear confinement in Lecture 24,¹ explain why when $\Delta_B \geq 5$, whenever $0 < |x| \leq \lambda n$ (for some $O(1)$ constant $0 < \lambda < 1$), then with high probability as $n \rightarrow \infty$, there exists a bit with $x_i = 1$ with the property that more than half of the checks it connects to, are not connected to any other bit in x . Use this observation to conclude that the greedy decoder succeeds with high probability on random LDPC codes when the error $|x| \leq \lambda n$.
- 15 **B:** In Lecture 25, we described how to build exponentially many deep local minima in the energy landscape of an LDPC code, far from any codeword, by considering syndromes with isolated violated checks sufficiently far from every other violated check. Taking inspiration from **A**, show that whenever $\Delta_B \geq 7$, the exponentially many local minima of the energy landscape of the LDPC code that we constructed in Lecture 25 exhibit 1-RSB.²

¹Hint: Take η close to $(\Delta_B - 2)/2$. How many checks connected to x are only connected to a single bit in x ? Find a simple (but useful) lower bound.

²Hint: Let x_0 be a local minimum where all violated checks are sufficiently far apart. We wish to show that this is 1-RSB. Suppose there was another local minimum nearby at $x_0 + y$ with $|y| \ll n$, and find a contradiction.

Problem 3 (Thermal topological order): In this problem we will show that the thermal state of the 4d toric code has topological order, in a sense we will make precise.

Before doing this, however, we need to introduce a useful technical tool, known as the **dressed logical algebra**. Suppose that $\mathbf{s}$ represents the global syndrome, and let $P_{\mathbf{s}}$ denote a projector onto the subspace of Hilbert space with syndrome $\mathbf{s}$. Let $Q_{\mathbf{s}}$ denote a set of Pauli strings that obey $Q_{\mathbf{s}_1} Q_{\mathbf{s}_2} = Q_{\mathbf{s}_1 + \mathbf{s}_2}$, and take $Q_{\mathbf{0}} = \mathbb{I}$, so that no product of $Q_{\mathbf{s}}$ with a trivial syndrome is a logical operator. (You can smartly choose small weight errors for easily correctable syndromes, to ensure you don't introduce artificial logical operations in this process). Now, given a logical representative operator L , define the scalar

$$\eta_{\mathbf{s}}(L) := \begin{cases} +1 & LQ_{\mathbf{s}} = Q_{\mathbf{s}}L \\ -1 & LQ_{\mathbf{s}} = -Q_{\mathbf{s}}L \end{cases} . \quad (1)$$

Then define the dressed logical

$$\bar{L} := \sum_{\text{admissible } \mathbf{s}} \eta_{\mathbf{s}}(L) L P_{\mathbf{s}}. \quad (2)$$

Here we have some freedom to choose what we mean by “admissible” syndrome: in this problem, for the 4d toric code, we will take it to correspond to a syndrome in which all closed loops of violated checks have length $\leq r$ (where we will choose a suitable value of r later).

- 10 **A:** Suppose that L_1 and L_2 are stabilizer-equivalent representatives of the same logical. Show that

$$\bar{L}_1 \bar{L}_2 = \sum_{\text{admissible } \mathbf{s}} P_{\mathbf{s}}. \quad (3)$$

More generally, argue that we can interpret $\bar{L}$ as a “Pauli-frame-tracked” logical operator for correctable errors: we can use $\bar{L}$ to logically manipulate information, rather than correcting all of the errors before applying an (ideal) logical gate.

For the purposes of this problem, we say that a mixed state has topological order if it cannot be prepared from an unentangled state in finite depth using a spatially local circuit. In this problem, you may use the following fact without proof: if I have two extended operators O_1 and O_2 that do not touch any qubits closer than $O(L)$ to each other, then if ρ can be prepared with a finite depth circuit,

$$\text{tr}(\rho O_1 O_2) = \text{tr}(\rho O_1) \text{tr}(\rho O_2). \quad (4)$$

- 20 **B:** Take $O_{1,2}$ to be widely separated logical representatives of the same logical operator, using the dressed logical algebra. Show that in the 4d toric code's thermal state, at sufficiently low temperatures, (4) cannot hold. To do this, choose the cutoff scale r in the dressed logical algebra above to scale with as $O(L)$, and show that in the thermal state, the right hand side of (4) is zero while the left hand side is non-zero. You'll want to use relevant facts about the 4d toric code's thermal state that were established in Lecture 22.
- 10 **C:** If you tried to apply this method to the 2d toric code, explain why it would not allow you to prove there is thermal topological order. In fact, give a simple heuristic argument why we do not expect thermal topological order in the 2d toric code.