Homework 1

Due: February 6 at 11:59 PM. Submit on Canvas.

Problem 1 (Microscopic model of a random walk): In Lectures 1 and 2 we discussed various cartoons for a random walk. These theories can be thought of as the effective theory for a more microscopic model, one of which we will consider in this problem: a "free particle" in one dimension which can exchange kinetic energy with a thermal bath. Consider a particle of mass m with position x and momentum p, with MSR Lagrangian (here γ and η are constants)

$$L = \pi\dot{x} + \sigma\dot{p} - \pi\frac{p}{m} + \sigma\left(\mathrm{i}\gamma\sigma + \eta p\right). \tag{1}$$

10 **A:** Give a clear physical interpretation of this theory.

A1. What would be the naive (noise-free) equations of motion? What is the noise?

A2. What is the Itō Fokker-Planck equation for this stochastic theory?

10 **B:** Let us now discuss the symmetries of this theory.

B1. Argue that this MSR Lagrangian is time-reversal symmetric with

$$\Phi = \beta\frac{p^2}{2m}. \tag{2}$$

What constraint on β must be obeyed?

B2. Identify any strong and weak continuous symmetries of the theory.

10 **C:** This problem is simple enough that we can actually compute explicitly the momentum two-point correlation function (take averages in the stationary state)

$$C_p(t) = \langle p(t)p(0)\rangle. \tag{3}$$

Show that for some constant K (which you need to fix):[1]

$$C_p(t) = K\mathrm{e}^{-\eta|t|}. \tag{4}$$

15 **D:** Let us now understand the transition to the random walk effective theory.

D1. Calculate the position correlation function

$$C_x(t) = \left\langle [x(t) - x(0)]^2 \right\rangle. \tag{5}$$

D2. In what limit does this answer reproduce the predictions of Lectures 1 and 2? Does the conclusion make physical sense given the microscopic model?

[1] *Hint:* This is probably easiest to do using the Langevin picture!

20 **Problem 2 (Planetary orbit):** Consider a planet of mass m interacting with a sun of mass $M \gg m$. For simplicity you can treat the sun as stationary at $(x, y, z) = (0, 0, 0)$ and restrict to studying just the motion of the planet directly. The classical Hamiltonian phase space for the dynamics thus has six coordinates, which you can label as (x_i, p_i) for $i = x, y, z$.

1. If the planet and sun interact via Newtonian gravity, what is the classical Hamiltonian for this system? You should treat the dynamics as non-relativistic.
2. Write down the MSR Lagrangian for this dissipationless Hamiltonian dynamics.
3. Because the planet has internal dynamics, it is possible for energy not to be conserved; however no net torques can act on the orbiting planet. Based on this fact, describe the most general possible time-reversal-symmetric Gaussian noise that can be added to the system. For simplicity, you can continue to use the MSR formalism.
4. For planetary sized objects, we might assume that if $\Phi = \beta H$, then $\beta \to \infty$ is a sensible limit (noise will be negligible). Describe qualitatively what will happen at late times for generic initial conditions.

Problem 3 (Thermal activation energy): A common physical argument is that the time it will take for a particle to hop over an energetic barrier of height U scales as

$$\tau \sim \exp[U/T] \tag{6}$$

at temperature T.

In this problem we will use our dissipative effective theories for thermal systems to justify this claim. For simplicity, we focus on an (overdamped) one-dimensional degree of freedom x with potential energy $V(x)$; we saw in Lecture 3 that the Fokker-Planck equation for such a system was:

$$\partial_t P = \partial_x \left[\gamma(x) \left(\partial_x + \beta V'(x)\right) P\right], \tag{7}$$

for some generic $\gamma(x)$. Here, you may take γ to be a constant for simplicity.

10 **A:** Since (7) is linear, we can find its general solution by finding a Green's function for initial condition $P(x, 0) = \delta(x - x')$. The resulting Green's function is denoted with

$$P(x, t|x', 0) = \langle x|\mathrm{e}^{-\hat{W}t}|x'\rangle, \tag{8}$$

using the notation from Lecture 3.

By performing some *quick* manipulations on (8), show that $P(x, t|x', 0)$ obeys both (7) and the **backward Fokker-Planck equation**

$$\partial_t P = \left(\partial_{x'} - \beta V'(x')\right) (\gamma \partial_{x'} P). \tag{9}$$

15 **B:** The backward Fokker-Planck equation is useful because it is well-suited to answer the following question: how long does it take for a particle to leave interval $S = [-a, a]$? The reason this problem is subtle is that at time τ, we should only keep track of the trajectories which obey $x(t) \in S$ for all $0 < t < \tau$. A clever mathematical way to do this is to find the special solution G_0 to (9) subject to the boundary conditions

$$G_0(\pm a, t > 0|x', 0) = 0, \tag{10a}$$
$$G_0(x \in S, 0|x', 0) = \delta(x - x'). \tag{10b}$$

We can intuitively think of this as *removing* a particle whenever it hits $x = \pm a$, thus keeping only the particles that stay trapped inside S in the remainder of the dynamics.

B1. The solution G_0 will not be a well-defined probability distribution on x (which should make sense given the intuitive argument above). Nevertheless, explain why

$$G(x', t) = \int_{-a}^{a} \mathrm{d}x \; G_0(x, t|x', 0) \tag{11}$$

can be interpreted as the probability that a particle starting at x' has not escaped after time t.[2]

B2. Explain why the average time τ it takes for a particle that starts at x' to hit the boundary is:

$$\tau(x') = -\int_{0}^{\infty} \mathrm{d}t \; t\partial_t G(x', t). \tag{12}$$

B3. Show that

$$\left(\partial_{x'} - \beta V'(x')\right) \gamma \partial_{x'} \tau(x') = -1. \tag{13}$$

C: To find a closed form answer, consider the potential

$$V(x) = \begin{cases} U|x|/a & |x| \le a \\ U(2a - |x|)/a & x > a \end{cases}. \tag{14}$$

Assume the particle starts at $x = 0$. Do you reproduce (6), or something "like it", when calculating the time for the particle to escape S?

Problem 4 (Long-range random walk): Consider a random walking particle where

$$x_n = \sum_{k=1}^{n} z_k, \tag{15}$$

where the z_k are independent and identically distributed random variables with probability density

$$\mathrm{p}(z) = \frac{\Gamma(\frac{\alpha}{2})}{\sqrt{\pi}\Gamma(\frac{\alpha-1}{2})} \frac{a^{\alpha-1}}{(x^2 + a^2)^{\alpha/2}}. \tag{16}$$

Assume that $\alpha > 1$, so that (16) is well-defined as a probability distribution.

Let τ be the time step between random walks, such that $x_n = x(n\tau)$. We want to take a continuum limit where $a \to 0$ and $\tau \to 0$.

1. Show that a formal Fokker-Planck equation is

$$\partial_t P(x, t) = \int_{-\infty}^{\infty} \mathrm{d}y \; w(y) P(x - y, t). \tag{17}$$

What should $w(y)$ be?

2. Discuss the interpretation of (17), given (16). In particular, are there values of $\alpha < \infty$ for which the Fokker-Planck equation will (approximately) reduce to the one for strictly local random walks, discussed in Lecture 2? How should a scale with τ so that the continuum limit of (17) is well-behaved?

[2] *Hint:* For $|x| < a$, G_0 obeys the ordinary Fokker-Planck equation as well. Show that $\partial_t G$ is captured wholly by boundary terms, and thus reach the desired conclusion.