Homework 2

Due: February 20 at 11:59 PM. Submit on Canvas.

15 **Problem 1 (Cell size):** Model a single-celled bacterial organism as a sphere of radius R (it lives in our three spatial dimensional universe). The bacterium requires oxygen to power the molecular machinery (power proteins, and so on). Assume that the bacterium must consume oxygen at a constant rate γ per unit volume. If the diffusion constant for oxygen dissolved in water is D and is the same inside and outside the cell, we can then model the concentration of dissolved oxygen around the cell via the modified diffusion equation:

$$\partial_t c = D\nabla^2 c - \begin{cases} \gamma & R > r \\ 0 & R < r \end{cases}, \tag{1}$$

1. Assume that the concentration of dissolved oxygen far away from the cell is c_0. Find a rotationally-invariant, time-independent solution to (1) in spherical coordinates, with $c(r \to \infty) = c_0$ and regularity at $r = 0$.
2. Argue that if the cell is too large (R is too large), then a physical solution to the diffusion equation does not exist. Deduce the maximal size of a cell $R_{\max}$.
3. Most single-celled organisms in nature (without very oblong shapes) have $R \sim 10^{-6}$ m, and e.g. E. coli bacteria consume oxygen at rate $\gamma \sim 20$ mM/s. For dissolved oxygen in water at room temperature, we measure $c_0 \sim 0.2$ mM and $D \sim 2 \times 10^{-9}$ m^2/s. Based on these numbers, do you think that the size of these single-celled organisms might be limited by their need to consume oxygen based solely on molecular diffusion?

Problem 2: Consider a theory in one spatial dimension with n conserved charges ρ^a ($a = 1, \dots, n$), which should be invariant under the following generalized time-reversal transformation:

$$\rho^a(x,t) \to C^{ab}\rho^b(x,-t). \tag{2}$$

Suppose that the stationary state is characterized by

$$\Phi = \int \mathrm{d}x \frac{1}{2}\rho^a(x) K^{ab} \rho^b(x). \tag{3}$$

Here K^{ab} is a symmetric, positive semi-definite matrix.

15 **A:** Build the most general possible hydrodynamic effective field theory.

- A1. What are the requirements on C^{ab} for this to be a valid generalized time-reversal transformation?
- A2. If π^a are the conjugate fields to ρ^a in the MSR Lagrangian, write down the generalized time-reversal transformation on π^a.

A3. Deduce that, keeping track only of quadratic order terms in π^a and ρ^a, the most general effective field theory is

$$\mathcal{L} = \pi^a \partial_t \rho^a + \mathrm{i} M^{ab} \partial_x \pi^a \partial_x \left(\pi^b - \mathrm{i}\mu^b \right). \tag{4}$$

List all possible constraints that you can think of on M^{ab}.

15 **B:** Consider the case $n = 2$, and take the matrix $C^{ab} = (\sigma^z)^{ab}$, with $K^{ab} = K_0 \delta^{ab}$.

B1. What is the most general possible form of M^{ab}?

B2. Find the dispersion relations for the hydrodynamic modes, and compare to the theory of ordinary diffusion. For convenience, you may assume that $M^{11} = M^{22}$ to simplify some algebra.

10 **C:** There are two senses in which our hydrodynamic effective field theory must be stable.

C1. Firstly, M^{ab} must be a positive-semidefinite matrix. Why?

C2. Secondly, the quasinormal modes must have $\mathrm{Im}(\omega(k)) \leq 0$. Why?

C3. Given generic positive-semidefinite matrices M and K, will the criterion of C2 always be satisfied? Why or why not?

Problem 3 (Fluctuating electrodynamics): Consider the theory of electrodynamics in some ambient "thermal" medium (e.g. a material at room temperature). We anticipate that due to the interactions between the material and electrodynamics, we will need to modify Maxwell's equations to account for the additional dissipation within any effective field theory.

15 **A:** To begin, we need to reformulate Maxwell's electrodynamics as an MSR Lagrangian. It will prove useful to begin with the Lagrangian for electromagnetism, gauge fixed so that $A_t = 0$:

$$\mathcal{L} = \frac{\epsilon}{2} \partial_t A_i \partial_t A_i - \frac{1}{4\mu} \left(\partial_i A_j - \partial_j A_i \right) \left(\partial_i A_j - \partial_j A_i \right).$$

A1. We can define a Hamiltonian theory by introducing the conjugate momentum D_i to the field A_i, which is equal to $\partial \mathcal{L} / \partial(\partial_t A_i)$. What is the physical interpretation of this conjugate momentum field?

A2. Find the Hamiltonian density

$$\mathcal{H} = D_i \partial_t A_i - \mathcal{L}(D_i, A_i). \tag{5}$$

A3. Suppose that the stationary state is thermal, such that

$$\Phi = \beta H = \beta \int \mathrm{d}^3 x \, \mathcal{H}. \tag{6}$$

Write down the dissipationless MSR Lagrangian.

A4. Explain how time-reversal symmetry should act in this system, and confirm that the MSR Lagrangian is time-reversal-symmetric.

20 **B:** Now we can consider incorporating dissipation into our theory. There are no explicit conservation laws we need to worry about.

B1. Write down the most important (i.e. fewest derivatives) dissipative terms within effective field theory; continue to respect time-reversal symmetry.

B2. Neglecting noise in the equations of motion, deduce the quasinormal mode spectrum for electromagnetism in a thermal medium. For this calculation, do not take the long wavelength limit.

B3. The effective field theory you have just described is an excellent model for electrodynamics in a particular kind of ambient medium. Explain what medium this is and why.

C: Show that you may directly integrate out fields[1] in the MSR Lagrangian from **B** to obtain the following effective field theory, which contains only the slow degrees of freedom:

$$\mathcal{L} = \pi_i \partial_t A_i + \mathrm{i}\kappa \pi_i \left(\pi_i - \partial_j \left(\partial_j A_i - \partial_i A_j\right)\right). \tag{7}$$

Here $\kappa > 0$ is a constant. Confirm that in the long-wavelength limit this theory is consistent with your answer to B2.

Problem 4 (Modulated symmetry): Suppose that instead of having a conserved charge density integrated over space, we consider instead the following modified conservation law (for constant $k_0 \neq 0$):

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}x \; \rho(x) \cos(k_0 x) = 0. \tag{8}$$

1. Deduce the hydrodynamic effective field theory for such a system, assuming Φ takes the same form as in Lecture 5. Demand that the hydrodynamic theory has emergent translation invariance.
2. Describe the quasinormal modes, and compare them to the conventional theory of diffusion.

[1] This means you can solve the Euler-Lagrange equations for certain fields (that show up at most quadratically), and plug those equations of motion back into the Lagrangian, to get a Lagrangian that depends on fewer fields.